BUCKLING OF THIN ELASTIC AND INELASTIC PLATES

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By

S. V. KRISHNA MOHAN RAO

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dedicated to

my beloved sister

SUJATA

CERTIFICATE

This is to certify that the research carried out by S.V. Krishna Mohan Rao for the preparation of the thesis, 'Buckling of Thin Elastic and Inelastic Plates', has been supervised by me. This thesis is being submitted to the Department of Civil Engineering, Indian Institute of Technology, Kanpur, in partial fulfillment of the requirements for the Degree of Master of Technology, and has not been submitted for a degree elsewhere.

July, 1981

Ashwini 8.7.81

(ASHWINI KUMAR)
Assistant Professor
Department of Civil Engineering
Indian Institute of Technology, Kanpur

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NOMENCLATURE

All symbols are defined whenever they appear first in the text.

a	-	length of plate in the current state
a_o	-	initial length of plate
a_1, a_2, a_3	-	principal stretches
b	-	width of plate in the current state
b_o	-	initial width of plate
$c_{11}, c_{12} \cdot\cdot$	-	moduli in eq.(3.2)
$\bar{c}_{11}, \bar{c}_{12} \cdot\cdot$	-	moduli in eq.(3.4)
C_{ijkl}	-	instantaneous moduli in Chapter 2
D	-	determinent
E	-	Young's modulus
e	-	compressive strain
e_y	-	compressive strain at yield
$f(\ )$	-	probability density function (Appendix A)
F_M	-	Mises critical stress function
F_T	-	Tresca critical stress function
F_M^*	-	analytical minimum stress function for Mises
F_T^*	-	analytical minimum stress function for Tresca
$F(\ )$	-	cumulative probability density function(Appendix A)
G_1, G_2, G_3	-	shear moduli
h	-	rate of hardening
$h_{\alpha\beta}$	-	hardening matrix
J	-	stress invariant $(= a_1 a_2 a_3)$

K_c, K_s	-	probablistic critical stress functions(Appendix A)
$\bar{K}_c, \bar{K}_s$	-	mean values of critical stress function(Appendix A)
K_{ijkl}	-	moduli defined in eq. (3.20)
m	-	number of half-waves along length at buckling
m_{ij}	-	components of unit normal to yield surface
n_i	-	unit surface normal to the element
p	-	indeterminable stress parameter eq.(4.7)
P_f	-	probability of failure (Appendix A)
P_r	-	reliability
S	-	surface
s_{ij}	-	nominal stress tensor
$s(\ )$	-	standard deviation of the variable (Appendix A)
t	-	thickness of the plate in the current state
t_o	-	initial thickness of the plate
T_j	-	surface traction
V	-	volume
$\bar{V}$	-	coefficient of variation (Appendix A)
v_i	-	velocity components in x_i directions
W	-	Ogden's strain energy function
u, v, w	-	velocity components along x,y,z direction
x_1, x_2, x_3	-	coordinate system in suffix notation
z	-	term indicating probability of failure(Appendix A)
α	-	Ogden's parameter in strain energy function
α	-	parameter in the stress-strain relation eq.(3.45)
γ	-	shear rate

η	-	coupled hardening parameter in Chapter 3
θ	-	E/2h
μ	-	shear modulus parameter in eq. (4.5)
μ	-	shear modulus in Chapter 3
ν	-	Poisson's ratio
σ	-	true (Cauchy) stress
σ_y	-	yield stress
σ^*	-	a parameter defined by eq. (3.49)
ϵ_{ij}	-	strain-rate tensor
ω_{ij}	-	rotation tensor
λ	-	mb/a (Appendix A)
$(\dot{})$	-	material derivative
$\frac{D(\ )}{Dt}$	-	Jaumann derivative
$\frac{\delta(\ )}{\delta t}$	-	Oldroyd (oonvective) derivative

ABSTRACT

The problem of buckling in thin rectangular plates under uniaxial compression is studied when (i) the material is elastic-plastic following the Tresca yield criterion with coupled hardening between the yield surface facets which meet at a vertex; (ii) the material is elastically incompressible and develops anisotropy during deformation. In the elastic-plastic case it is shown that the yield surface corner considerably lowers the bifurcation stress, either for independent hardening or for coupled hardening, compared to that obtained by using the Mises yield criterion; the latter is known to give buckling stresses much higher than the experimental results. In elastic case, the stress at bifurcation is found for a material defined by the Ogden's strain energy function. Lastly, the inelastic buckling problem has been re-examined with a probablistic approach; the buckling stress has been calculated for a given probability of failure and for a desired variation in the material parameters and the plate dimensions.

CHAPTER 1

INTRODUCTION

1.1 BRIEF LITERATURE REVIEW

It is not the aim here to review a vast amount of literature available in the theory of stability of structures, mostly under 'small deformation' approximation. Related to the present work we concentrate only on the subject matter based on the continuum approach to the stability investigation.

A new trend in the subject of stability was induced due to Shanley's (1947) demonstration of the fact that the problems of bifurcation (i.e. non-uniqueness of solution) and that of instability are, in general, two different concepts though there is a definite relationship between each other. Some ten years later, Hill presented a general three-dimensional continuum theory of uniqueness and stability, and thereby established exact connections between the uniqueness of deformation and the stability of equilibrium configuration for elastic, rigid-plastic and elastic-plastic solids. These principles are also available in a compact form (Hill, 1959, 61).

In examining the problem of uniqueness of a body under finite strain, usually two methods have been

adopted. The first is the method of small deformations superimposed on finite deformations as available in, for example Green and Zerna(1954) and Biot (1965). A system of equations is obtained for the equilibrium of the incremental deformation; these equations together with rate boundary conditions and rate material properties constitute the rate problem. The second method makes use of a variational technique in which a certain functional of velocity gradients (or of stress-rates) vanishes in the absence of a unique solution; non-vanishing of the functional implies uniqueness. A detailed account of both the approaches is contained in Hill's work.

Solutions of some specific rate problems in elastic solids have been given by a number of investigators, most notably by Biot (1965). Plain strain/plane stress bifurcations in elastic-plastic plate were examined, amongst many, by Ramsey (1969), Dubey and Ariaratnam(1969), Hill and Hutchinson (1975), Young (1976) and Needleman (1979). Wilkes (1955), and later on Vaughan (1971) and Sierakowski et.al. (1975), obtained bifurcation conditions in solid elastic cylinders under end thrust, while instabilities in thin tubes were investigated by Dubey (1969) and Storåkers (1975). Recent studies by Haughton and Ogden (1979a,b) and Niyogi (1980) present a comprehensive treatment of instabilities in thin and thick-walled elastic cylinders

under axial compression. Bifurcations in elastic plastic plates under pure bending have recently been reported by Triantafyllidis (1980).

Hill and Sewell (1960 a,b) have applied the variational technique in an investigation of the buckling of an inelastic column. The analysis has been extended to the buckling of elastic-plastic plates also (Sewell 1963, 64). Some investigations in the stability of rigid-plastic cylindrical shells under pressure and/or axial load are available due to Storåkers (1971), Kumar and Ariaratnam (1975,76) and Kumar and Shukla (1981). The finite element formulation of Hill's variational principle has been successfully applied to study instabilities in cylindrical bars under tension and compression (Needlemen 1972, 73), bifurcation in plane tension test (Burke and Nix, 1979), bifurcation in cylindrical shell under internal pressure (Chu, 1979) and buckling of plates in the presence of cracks (Markström and Storåkers, 1980).

1.2 PRESENT WORK

The thesis presents an application of Hill's variational technique to examine stability in axially compressed thin elastic and elastic-plastic plates. There are large number of investigations available concerning buckling of plates in the inelastic range,

but most of them are merely an extension of the classical buckling theory in elasticity. The most notable contributions are due to Sewell (1963,64) who applied the continuum theory of uniqueness and stability to determine the buckling stress of an elastic plastic plate of Mises material. It was also shown that buckling stress could be sensitive to changes in the directions of the normals at the Tresca yield surface corner and that it could predict a value of the critical stress much lower than that for the Mises yield surface. Similar observations were also reported later on by Dubey and Ariaratnam(1969). The present investigation reexamines Sewells problem when the current state of stress is at the corner of Tresca yield surface. The material is supposed to obey Hill's (1966) constitutive relations for metal crystals deforming by multislip. Then, using the flow-rule at a vertex (Sewell, 1972, 74), it has also been possible to include the effect of coupled hardening between the facts of the Tresca yield surface that meet at a vertex. It is shown, by using a stress-strain curve of the Ramberg-Osgood type (or even otherwise) that a considerable reduction in the critical stress is possible by using the Tresca yield criterion instead of the Mises, which is known to give higher buckling stress as compared to the experimental values.

A probablistic approach, in brief, is also present~~age~~ed to study the buckling of elastic-plastic plates. Curves are drawn to obtain the buckling stress for a known probability of failure and for the desired variation in the material parameters and the plate dimensions.

Also, buckling of thin elastic plates has been reviewed by considering Ogden's(1972) strain-energy function for a general incompressible material. The material is initially isotropic and develops anisotropy due to deformation. The outline of the thesis is as follows:

Chapter two presents a brief account of Hill's sufficient conditions for the uniqueness of deformation and the stability of equilibrium configuration. The stability criterion is applied to an axially compressed thin elastic-plastic plate in Chapter three and to an elastic plate in Chapter four. The conclusions are summarised in Chapter five. Appendix A gives a probabilistic approach to determine the buckling stress of an inelastic plate. Appendix B gives the standard normal probability tables.

CHAPTER 2

GENERAL THEORY OF STABILITY AND UNIQUENESS

The process of continuing deformation of a finitely deformed solid is marked by large elastic or/ and plastic deformations terminating in non-uniqueness. Upto what extent the deformation of a structure is unique, under given loading conditions, can be determined by using some 'uniqueness condition' and when the deformation becomes non-unique, i.e. when a bifurcation occurs, can be found by solving the rate boundary value problem. In this chapter, we shall discuss the formulation of the rate problem and derive sufficient conditions for uniqueness of deformation and stability of equilibrium configuration.

2.1 RATE PROBLEM

It is usually convenient to represent the equations of equilibrium for the incremental deformation from some current equilibrium state in the rate form. Similarly, the boundary conditions and the constitutive relations for the incremental deformation can also be expressed in the rate form. The rate equilibrium equations together with the rate boundary conditions and the rate material properties constitute the rate problem.

2.1.1 Equilibrium Equations

Let the body in an initial configuration B^o be finitely deformed to some current configuration B. Suppose that in the current state, the traction T_j are specified on some part S_T of the surface S, i.e.

$$T_j = n_i \sigma_{ij} \tag{2.1}$$

and the displacements are prescribed on the remainder. Here, in (2.1) σ_{ij} is the true (Cauchy) stress and n_i is the unit normal to the element. Then, if the body force is f_j per unit volume in the current configuration, the equations of equilibrium are

$$\sigma_{ij,i} + f_j = 0 \tag{2.2}$$

where a comma denotes derivative with respect to x_i ; or alternatively can be expressed as

$$s_{ij,i} + f_j = 0 \tag{2.2a}$$

when the current state is taken as the reference state. In (2.2a) s_{ij} is the nominal (Lagrangian) stress.

For the analysis of the problem of bifurcation of equilibrium, we need to examine the behaviour of the body when it is perturbed from its deformed state B to a neighbouring state B'. During this incremental deformation, let the velocity v_i and the nominal traction-rates $\dot{T}_j$ be prescribed on part S_V and S_T respectively, of

the surface S of the body. The equations of continuing equilibrium in terms of the Lagrangian stress-rate are (Hill, 1959)

$$\dot{s}_{ij,i} = 0 \quad \text{in} \quad V \qquad (2.3)$$

neglecting body forces. Here, V is the current volume of the body. The material derivative of the true stress, $\dot{\sigma}_{ij}$ is related to $\dot{s}_{ij}$ by the following relation

$$\dot{s}_{ij} = \dot{\sigma}_{ij} + \sigma_{ij}\, v_{k,k} - \sigma_{jk}\, v_{i,k} \qquad (2.4)$$

when the material is incompressible (i.e. $v_{k,k} = 0$) and the current stress-distribution is homogeneous, the equilibrium equation can alternatively be expressed as

$$\dot{\sigma}_{ij,i} = 0 \quad \text{in} \quad V \qquad (2.3a)$$

Equations (2.3) are the most concise form of the rate equilibrium relations. They have, however, been expressed in various other forms in the literature. Some of the well-known ones have been discussed by Bazant (1971). For example, in terms of Jaumann flux of the Cauchy stress, we have

$$\left(\frac{D\,\sigma_{ij}}{Dt}\right)_{,i} + \left(\sigma_{ik}\,\omega_{jk} + \sigma_{ij}\,v_{k,k} - \sigma_{jk}\epsilon_{ik}\right)_{,i} = 0 \qquad (2.5)$$

as is available in Biot (1965). Here $\omega_{ij} = \frac{1}{2}(v_{i,j} - v_{j,i})$ is the antisymmetric part of the velocity gradient tensor

while $\varepsilon_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$ is the symmetric part. On the other hand, Green and Zerna (1954) and Wesolowski (1962) express their basic equations in terms of convected derivative of the Cauchy stress

$$\left(\frac{\delta\sigma_{ij}}{\delta t}\right)_{,i} + (\sigma_{ij}v_{k,k} + \sigma_{ik}v_{j,k})_{,i} = 0 \qquad (2.6)$$

2.1.2 Boundary Conditions

The rate boundary condition are normally written as

$$\dot{T}_j = n_i \dot{s}_{ij} \text{ on } S_T$$

$$v_i = V_i \quad \text{on} \quad S_V \qquad (2.7)$$

The quantity v_i is usually a given function of position on the boundary, while $\dot{T}_j$ (the nominal traction-rate) may depend on position and local unknown velocities. Various formsfor $\dot{T}_j$, involving configuration dependent loading, have been derived by Hill (1962) and Dubey(1970). For example, the dead loads are characterized by $\dot{T}_j = 0$.

2.1.3 Constitutive Relations

We consider only those elastic-plastic solids having no natural time (i.e. no creep or real strain-rate effects). The instantaneous (incremental) material behaviour ensuing from the given current state is expressed by relation of the form

$$\text{Stress-rate} = f\ (\text{strain-rate}) \qquad (2.8)$$

where f is a homogeneous function of degree one in the strain-rate components with the property $f(0) = 0$, but may be non-linear. At this stage we need not specify the forms of 'f' which will be done at a later stage.

2.2 UNIQUENESS CRITERION

For incremental deformation from the current configuration B to a neighbouring configuration B', we have relations (2.3) and (2.7). A condition for uniqueness of solution is obtained as follows. Suppose that there could be more than one solution of a boundary-value problem and let the difference between any two corresponding quantities be denoted by the prefix Δ. Then

$$\Delta \dot{s}_{ij,i} = 0 \text{ in } V \qquad (2.9)$$

$$\Delta \dot{T}_j = n_i \Delta \dot{s}_{ij} = 0 \text{ on } S_T \qquad (2.10)$$

$$\Delta v_j = 0 \text{ on } S_V \qquad (2.11)$$

Hence

$$\int_S \Delta \dot{T}_j \Delta v_j \, dS = \int_V \Delta \dot{s}_{ij} \Delta v_{j,i} \, dV = 0 \qquad (2.12)$$

by using Gauss's divergence theorem. A sufficient condition for uniqueness is, therefore, that

$$\int_V \Delta \dot{s}_{ij} \Delta v_{j,i} \, dV > 0 \qquad (2.13)$$

for all pairs of continuous differentiable velocity fields taking the prescribed values on S_V and the corresponding

stress-rates $\dot{s}_{ij}$ obtained from the v_i through the constitutive relation. The reverse inequality would also give a sufficient condition, but is not considered here because such states are unstable (shown later on). It is important to note that when the constitutive relations are linear, the integral in (2.13) is a single-valued functional of v_j (regarded as a single field), but when they are non-linear, it is multivalued.

2.3 STABILITY CRITERION

To be definite suppose that a part S_V of the surface of the body is rigidly constrained and that dead loads are maintained on the part S_T. The body, initially at rest, is imagined to be set in motion by some kind of perturbation . The initial state is said to be stable if the amplitude of the ensuing displacement is always vanishingly small whenever the perturbation itself is. If, on the other hand, the amplitude is finite for atleast one type of disturbance, however small this might be, the state is said to be unstable. Evidently, a sufficient condition for stability is that the internal energy stored should exceed the work done by the external loads during the resulting motion [cf., e.g. Hill (1959), Prager (1961)] .

In a time interval δt, due to a virtual displacement $v_i\ \delta t$, the nominal stress changes from s_{ij} to $(s_{ij}+\dot{s}_{ij}\delta t)$,

still referred to the current state. Therefore, the increase in the internal energy will be

$$\delta t \int_V (s_{ij} + \frac{1}{2} \dot{s}_{ij}\, \delta t)\, v_{j,i}\, dV$$

The virtual work done by the external forces during the time interval δt, under the load on S_T and rigid constraints on S_V, i.e. $v_j \equiv 0$ on S_v, is

$$\delta t \int_S T_j v_j dS = \delta t \int_S n_i s_{ij} v_j dS = \delta t \int_V s_{ij} v_{j,i} dV$$

using the divergence theorem and the current equilibrium condition $s_{ij,i} = 0$. Hence, to the second order in δt, the internal work minus the external work is

$$W = \frac{1}{2} (\delta t)^2 \int_V \dot{s}_{ij}\, v_{j,i}\, dV \qquad (2.14)$$

Now, suppose that K_0 is the initial infinitesimal kinetic energy of the body. After the time δt, in the new configuration, its energy is $K = K_0 - W$, by the principle of conservation of energy. Since K can never be negative, an upper bound for δt can be obtained using (2.14) if W remains positive. Therefore, the positive-definiteness of (2.14) is sufficient for stability, i.e. for stability we must have

$$\int_V \dot{s}_{ij}\, v_{j,i}\, dV > 0 \qquad (2.15)$$

for all continuous differentiable velocity fields which vanish on S_V (but are not identically zero) and are

compatible via. the constitutive equations with the stress rates $\dot{s}_{ij}$.

2.3.1 Stability vs. Uniqueness

Consider the relation between the conditions (2.13) and (2.15). We note that the Δv_j fields admitted in (2.13) are exactly those admitted in (2.15) since both vanish on S_v and are otherwise arbitrary. However, (2.15) is a single-valued functional while (2.13) is multi-valued when constitutive equations are non-linear and single-valued only when they are linear. Consequently, the conditions are identical only when the rate-equations are linear.

However, it may be mentioned here that the condition (2.13) is not necessary for uniqueness; the positive identification of bifurcation states involves actually demonstrating solutions to (2.9) - (2.11). For certain linear solids, a solution does exist in those states for which the functional in (2.13) is generally positive but vanishes for at least one non-zero field Δv_j. It is such bifurcations that would normally be reached first in any actual process of deformations since they terminate a continuous range of states which not only satisfy (2.13) but are also stable.

For the linear elastic-plastic solids, it is obvious now that the conditions for uniqueness and

stability are identical;for uniqueness we must have

$$\int \dot{s}_{ij}\, v_{j,i}\, dV > 0$$

Following Hill (1959), the linear rate constitutive equations can be expressed in terms of the Jaumann derivative of the Kirchhoff stress, $D\tau_{ij}/Dt$, as follows

$$\frac{D\tau_{ij}}{Dt} = C_{ijkl}\, \epsilon_{kl} \qquad (2.16)$$

such that $C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}$. Using the relation between $\dot{s}_{ij}$ and $D\tau_{ij}/Dt$, viz.

$$\dot{s}_{ij} = \frac{D\tau_{ij}}{Dt} + \sigma_{ik}\, v_{j,k} - \sigma_{ik}\, \epsilon_{jk} - \sigma_{jk}\, \epsilon_{ik} \qquad (2.17)$$

the uniqueness (stability) criterion takes the form

$$\int_V (C_{ijkl}\, \epsilon_{ij}\, \epsilon_{kl} - 2\sigma_{ij}\, \epsilon_{kj}\, \epsilon_{ki} + \sigma_{ij}\, v_{k,i}\, v_{k,j})\, dV > 0 \qquad (2.17)$$

It is clear that once the instantaneous moduli C_{ijkl} and the stresses σ_{ij} are regarded as having assigned values, the criterion (2.17) may be used to determine within what limits these values must lie in order to ensure uniqueness (stability).

Finally, like in the classical theory, certain restrictions are imposed on the 'current' C_{ijkl}; we suppose them to be such that

$$\frac{D\tau_{ij}}{Dt} e_{ij} = C_{ijkl}\, e_{ij}\, e_{kl} > 0 \qquad (2.18)$$

for all non-zero stress-rates. This enables one to avoid considering the possibility of bifurcation in a state of zero stress, which (2.17) might otherwise permit.

CHAPTER 3

INELASTIC PLATE BUCKLING

3.1 INTRODUCTION

The formulation developed in Chapter 2 is now applied to study the bifurcation of thin plates of elastic-plastic material. Unlike the case of elastic buckling, there have been only a few investigations related to the inelastic plate buckling, the most notable ones are those of Sewell(1963,64) wherein, for the first time, the continuum theory of bifurcation was applied. It was shown that the bifurcation stress (buckling stress) is very sensitive to the local yield surface normal; in particular the 'fan' of possible normals at a Tresca corner can give the buckling load markedly lower than that obtained using the Mises smooth yield surface. Similar observations have also been reported by Dubey and Ariaratnam (1969).

The present investigation is an extension of Sewell's (1964) work. The bifurcation stress is calculated for a simply supported plate in uniaxial compression when the stress is at a vertex of the Tresca yield surface. The effect of coupled hardening between the yield surface facts which meet at the vertex is included.

A considerable reduction in the bifurcation stress is found which is due to (i) the Tresca corner itself with no coupling and (ii) the coupled hardening.

3.2 FORMULATION OF THE PROBLEM

Consider a thin rectangular plate which in the current state has edges $x_1 = 0$, a and $x_2 = 0$, b and is bounded by plane faces $x_3 = \pm t/2$. The plate is simply supported on all edges. The current stress distribution is uniform uniaxial compression in the x_1 - direction i.e.,

$$\sigma_{ij} = -\sigma\, \delta_{i1}\, \delta_{j1} \tag{3.1}$$

with uniform $\sigma > 0$ (Fig. 3.1). Whatever the previous stress and strain history may have been, we suppose that the linear rate equations of the following type are given at each point.

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} D\tau_{11}/Dt \\ D\tau_{22}/Dt \\ D\tau_{33}/Dt \end{bmatrix} \tag{3.2}$$

and $2G_1\, \varepsilon_{23} = D\tau_{23}/Dt,\ 2G_2\, \varepsilon_{13} = D\tau_{13}/Dt,$

$$2G_3\, \varepsilon_{12} = D\tau_{12}/Dt \tag{3.3}$$

There is absent here not only a transverse but also in-plane shear interaction with the normal terms. It is assumed that the determinent D (say) of coefficients

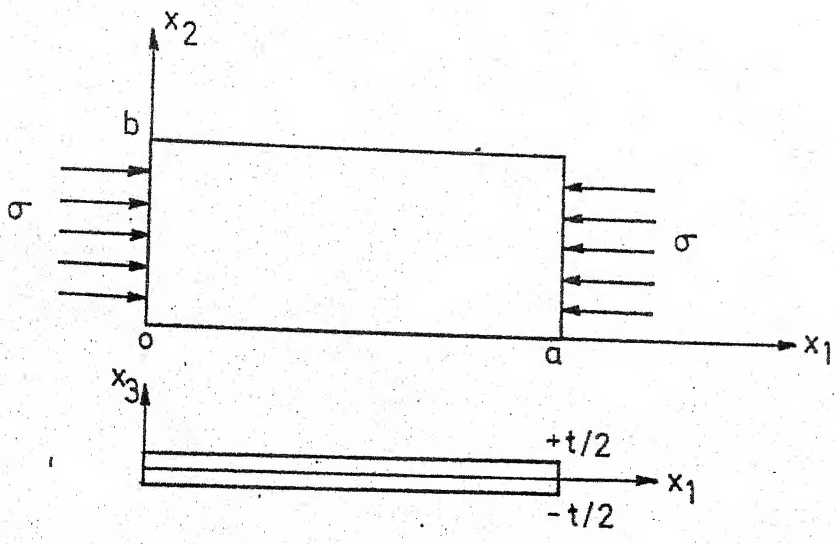

FIG.3.1 SIMPLY SUPPORTED PLATE UNDER COMPRESSION

in (3.2) is non-zero. Then, (3.2) has a unique inverse of the form (2.16) which we write as

$$\begin{bmatrix} D\tau_{11}/Dt \\ D\tau_{22}/Dt \\ D\tau_{33}/Dt \end{bmatrix} = \frac{1}{D} \begin{bmatrix} \bar{c}_{11} & \bar{c}_{12} & \bar{c}_{13} \\ \bar{c}_{12} & \bar{c}_{22} & \bar{c}_{23} \\ \bar{c}_{13} & \bar{c}_{23} & \bar{c}_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \end{bmatrix} \qquad (3.4)$$

where a 'bar' denotes the cofactor of the associated element in (3.2). For (2.18) to hold, we will have to ensure that

$$\begin{array}{lll} c_{11} > 0, & c_{11}\, c_{22} - c_{12}^2 > 0, & D > 0 \\ G_1 > 0, & G_2 > 0, & G_3 > 0, \end{array} \qquad (3.5)$$

Further, the plane faces $x_3 = \pm\ t/2$ are free of any applied traction -rates; this means that

$$\dot{s}_{33} = \frac{D\tau_{33}}{Dt} = 0$$

$$\dot{s}_{31} = \frac{D\tau_{13}}{Dt} + \sigma\, \varepsilon_{13} = 0 \qquad \text{at } x_3 = \pm\ t/2 \tag{3.6}$$

$$\dot{s}_{32} = \frac{D\tau_{23}}{Dt} = 0$$

The first of (3.6) is satisfied by actually imposing $D\tau_{33}/Dt = 0$ throughout the plate thickness. This condition may be written as

$$\bar{c}_{13}\, \varepsilon_{11} + \bar{c}_{23}\, \varepsilon_{23} + \bar{c}_{33}\, \varepsilon_{33} = 0 \tag{3.7}$$

Therefore, the corresponding 'reduced' constitutive equations relating the inplane normal stress-rate and strain-rates are

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{bmatrix} \begin{bmatrix} D\tau_{11}/Dt \\ D\tau_{22}/Dt \end{bmatrix} \tag{3.8}$$

and

$$\begin{bmatrix} D\tau_{11}/Dt \\ D\tau_{22}/Dt \end{bmatrix} = \begin{bmatrix} c^*_{11} & c^*_{12} \\ c^*_{12} & c^*_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \end{bmatrix} \tag{3.9}$$

where

$$c^*_{11} = \frac{c_{22}}{c_{11}\, c_{22} - c_{12}^2}, \quad c^*_{12} = -\frac{c_{12}}{c_{11}\, c_{22} - c_{12}^2},$$

$$c^*_{22} = \frac{c_{11}}{c_{11}\, c_{22} - c_{12}^2} \tag{3.10}$$

and (3.5) imply that

$$c^*_{11},\ c^*_{22}\ ,\ G_3 > 0\ ;\ c^*_{11}\ c^*_{22} - c^{*2}_{12} = \frac{1}{c_{11}\ c_{22} - c^2_{12}} > 0 \qquad (3.11)$$

The last two of (3.6) can be taken care of by requiring that e_{12} and e_{23} are small (not necessarily zero) compared to e_{11} and e_{22}. We now select a class of velocity field which satisfy (3.7) and also incorporate the usual engineering assumption that the line elements perpendicular to the middle surface incipiently remain straight and perpendicular to it. This class is (Sewell, 1963)

$$v_1 = u - x_3 \frac{\partial w}{\partial x_1}$$

$$v_2 = v - x_3 \frac{\partial w}{\partial x_2}$$

$$v_3 = w - x_3 \left(\frac{\bar{c}_{13}}{\bar{c}_{33}} \frac{\partial u}{\partial x_1} + \frac{\bar{c}_{23}}{\bar{c}_{33}} \frac{\partial v}{\partial x_2} \right) + \frac{1}{2} x_3^2 \left(\frac{\bar{c}_{13}}{\bar{c}_{33}} \frac{\partial^2 w}{\partial x_1^2} + \frac{\bar{c}_{23}}{\bar{c}_{33}} \frac{\partial^2 w}{\partial x_2^2} \right) \qquad (3.12)$$

Here u, v, w are arbitrary functions of x_1 and x_2 representing the velocity components of the middle surface of the plate.

3.3 APPLICATION OF THE UNIQUENESS CRITERION

The next few steps are similar to those used in elastic plate buckling analysis. Here, we essentially follow Sewell (1963) and do not reproduce several routine calculations. The stress distribution (3.1) and the velocity field (3.12) are inserted in the uniqueness/stability criterion (2.17). The effect of transverse shear stiffening (i.e., G_1 and G_2 terms) is nearly negligible. In the minimisation process of (2.17) it can be shown that (Sewell, 1963) the best choice of functions $u\ (x_1,x_2)$ and $v\ (x_1,x_2)$ is just $u \equiv v \equiv 0$ (apart from a possible rigid body rotation). Therefore, it remains to minimise with respect to transverse velocity field $w(x_1,x_2)$ of the simply supported middle surface. Then, interms of $w(x_1,x_2)$, the uniqueness criterion becomes

$$\frac{t^3}{12}\int_0^a\int_0^b \Big[\Big\{c^*_{11}\Big(\frac{\partial^2 w}{\partial x_1^2}\Big)^2 + 2\,c^*_{12}\,\frac{\partial^2 w}{\partial x_1^2}\,\frac{\partial^2 w}{\partial x_2^2} + c^*_{22}\Big(\frac{\partial^2 w}{\partial x_2^2}\Big)^2 + 4G_3\Big(\frac{\partial^2 w}{\partial x_1\,\partial x_2}\Big)^2\Big\} - \sigma\Big\{\frac{12}{t^2}\Big(\frac{\partial w}{\partial x_1}\Big)^2 - \Big(\frac{\partial^2 w}{\partial x_1^2}\Big) - \Big(\frac{\partial^2 w}{\partial x_1\partial x_2}\Big)^2\Big\}\Big]\,dydx > 0 \qquad (3.13)$$

Satisfying all the boundary conditions, the transverse velocity field can be taken in the form

$$w = \sin \frac{m\pi x_1}{a} \quad \sin \frac{m\pi x_2}{b} \tag{3.14}$$

where m and n are positive integers. Use of (3.14) reduces (3.13) to the form

$$[(\frac{m}{a})^4 \; c^*_{11} + (\frac{mn}{ab})^2 \; 2 \; (c^*_{12} + 2 \; G_3) + (\frac{n}{b})^4 \; c^*_{22}] -$$

$$\frac{12}{\pi^2 t^2} (\frac{m}{a})^2 \sigma [1 - (\frac{mt}{a})^2 \frac{\pi^2}{12} - (\frac{nt}{b})^2 \frac{\pi^2}{12}] > 0 \tag{3.15}$$

In addition to (3.11) we also require that

$$c^*_{12} + 2 \; G_3 \geq 0 \tag{3.16}$$

It follows that, for any given m and σ , the left side of (3.15) is least when n = 1, i.e., there is only one half-wave perpendicular to the loading direction. Also, the thin-plate assumption obviates the need of retaining the terms involving $(t/a)^2$ and $(t/b)^2$ in (3.15); consequently it is sufficient to express the bifurcation stress formula ss

$$\sigma = \min \frac{\pi^2 t^2}{12b^2} [(\frac{mb}{a})^2 \; c^*_{11} + 2 \; (c^*_{12} + 2 \; G_3) + (\frac{a}{mb})^2 \; c^*_{22}] \tag{3.17}$$

It remains to identify the coefficients of the constitutive equation (3.2) so that once they are known, the right side of (3.17) can be minimised (for a given b/a) with

respect to the integer 'm'. However if we relax[+] the constraint that m be integral, then minimisation can actually be carried with respect to the continuous 'm' and the minimum is attained at

$$\frac{mb}{a} = \left(\frac{c^{*}_{22}}{c^{*}_{11}} \right)^{1/4} \qquad (3.18)$$

Then the corresponding bifurcation stress formula is

$$\sigma = \frac{\pi^2 t^2}{6 b^2} \left[c^{*}_{12} + 2 G_3 + \sqrt{c^{*}_{11} c^{*}_{22}} \right] \qquad (3.19)$$

3.4 CONSTITUTIVE RELATIONS FOR ELASTIC-PLASTIC MATERIALS

Here, we refer to Hill's (1966) constitutive relations for incremental deformation of metal crystals by multislip. These relations take into account the cross coupling between glide and hardening on different slip-systems.

The incremental deformation in a crystal is treated as the sum of contributions from two independent atomic mechanisms on a macroscopic scale;these contributions are (i) an overall elastic distortion of the lattices and (ii) plastic simple shears that do not disturb the lattice geometry. The shears have specific planes and directions each associated pair of which defines a

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+ can be justified in many cases as shown in Sec. 3.5.

'glide system'. At each stage of a loading programme the particular 'active' systems are considered to be uniquely determined by the current stress-distribution; its shearing component in a direction of possible glide must attain a critical positive value that depends on the history of overall straining and on the particular system. During incremental deformation each system hardens at a definite rate which may be a function of every rate of shear, typically γ say.

In an elastic element we take the rate equations to be

$$\epsilon^{e}_{ij} = K_{ijkl} \frac{D\tau_{kl}}{Dt} \tag{3.20}$$

where the moduli have the symmetry of C_{ijkl} in (2.16):

$$K_{ijkl} = K_{jikl} = K_{ijlk} = K_{klij} \tag{3.21}$$

The plastic part of the overall strain rate ϵ_{ij} is expressed as

$$\epsilon^{p}_{ij} = \gamma_{\alpha}\, m_{ij\alpha} \tag{3.22}$$

Greek suffixes will range over 1 to N, for some specified N, with summation implied when they are repeated unless otherwise mentioned. Each γ_{α} is interpretable as a shear rate in a direction with unit vector components $t_{i\alpha}$ on a glide plane with unit normal components $n_{j\alpha}$, with

$$m_{ij\alpha} = \frac{1}{2}\,(t_i\, n_j + t_j\, n_i)_\alpha \qquad (3.23)$$

Thus $m_{ij\alpha}\;\; m_{ij\alpha} = \frac{1}{2}$ (no sum on α) $\qquad$ (3.24)

Accordingly to Sewell (1971) the precise single crystal interpretations do not have to be taken over directly. Instead, the connection with more classical plasticity can associate the N tensors $\sqrt{2}\, m_{ij\alpha}$ with the unit normals to the facets of a singular yield surface having a vertex at the given stress point. We shall illustrate N = 2 with the Tresca criterion. When N = 1, equations appropriate to a smooth yield surface can be recovered. Following Hill (1966) and Sewell (1971), the right hand side of (3.22) is expressed in the form

$$\gamma_\alpha\, m_{ij\alpha} = h^{-1}_{\alpha\beta}\,\left(\frac{D\,\tau_{kl}}{Dt}\, m_{kl\beta}\right) m_{ij\alpha}$$

in which the $h_{\alpha\beta}$ are the elements of a given N x N 'hardening' matrix such that

$$h_{\alpha\beta} \quad \text{positive definite} \qquad (3.25)$$

Finally the constitutive relations for an elastic plastic element become

$$\epsilon_{ij} = (K_{ijkl} + h^{-1}_{\alpha\beta}\, m_{ij\alpha}\, m_{kl\beta})\,\frac{D\tau_{kl}}{Dt} \qquad (3.26)$$

This is the form of the constitutive law from which we can expect to identify coefficients in (3.2) and (3.3).

We adopt the Tresca yield criterion which, in the plane $\sigma_{33} = 0$ of principal stress space, is shown in Fig. 3.2. We assume that the principal axes of m_{ij1} and m_{ij2} coincide with those of stress. Then it follows that,

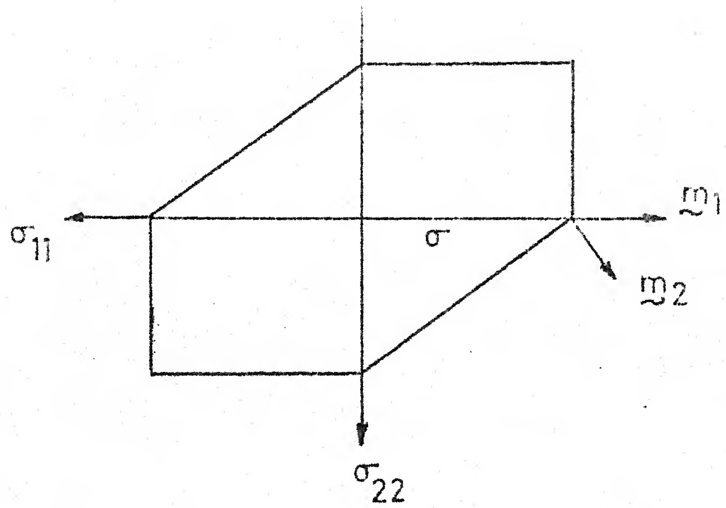

FIG.3.2 TRESCA YIELD SURFACE

$$m_{ij1} = \frac{1}{2}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad m_{ij2} = \frac{1}{2}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.27}$$

We further assume that the elastic part of the material response is isotropic with Young's Modulus E and Poisson's ratio ν. Then $\mu = E/2\,(1+\nu)$ is the positive elastic shear modulus. Further, the 2x2 assigned hardening

matrix is written as

$$h_{\alpha\beta} = h \begin{bmatrix} 1 & \eta \\ \eta & 1 \end{bmatrix} \tag{3.28}$$

in which

$$h > 0 \, , \quad -1 < \eta < 1 \tag{3.29}$$

in order to satisfy (3.26). The parameter η represents the amount of 'coupled hardening'. It may be mentioned here that in the investigation by Sewell (1964), the hardening parameter h is twice the present 'h' when $N = 1$ and the yield surface normal is $\sqrt{2}m_1$. So far as is known, only three types of hardening matrix have ever been proposed, and all are special cases of (3.28):

(i) $\eta = 1$, isotropic hardening (Taylor)
(ii) $\eta = 0$, independent hardening (Koiter)
(iii) $\eta = 1/2$, kinematic hardening (Budiansky and Wu).

The relations (3.27) and (3.28) and the isotropic form of K_{ijkl} are now substituted into (3.25). The constitutive equations then take the following form

$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 + \frac{\theta}{1+\eta} & -\nu - \frac{\theta}{2(1+\eta)} & -\nu - \frac{1}{2(1+\eta)} \\ -\nu - \frac{\theta}{2(1+\eta)} & 1 + \frac{\theta}{2(1-\eta^2)} & -\nu - \frac{\eta\theta}{2(1-\eta^2)} \\ \nu - \frac{1}{2(1+\eta)} & -\nu - \frac{\eta\theta}{2(1-\eta^2)} & 1 + \frac{\theta}{2(1-\eta^2)} \end{bmatrix} \begin{bmatrix} \frac{D\tau_{11}}{Dt} \\ \frac{D\tau_{22}}{Dt} \\ \frac{D\tau_{33}}{Dt} \end{bmatrix} \tag{3.30}$$

and $2\mu\, e_{23} = \frac{D\tau_{23}}{Dt}$, $2\mu\, e_{13} = \frac{D\tau_{13}}{Dt}$, $2\,\mu\, e_{12} = \frac{D\tau_{12}}{Dt}$

where $\theta = E/2h$. From (3.30) are can easily calculate the starred moduli using (3.10):

$$c^*_{11} = \frac{E}{D}\left(1 + \frac{\theta}{2(1-\eta^2)}\right)$$

$$c^*_{12} = \frac{E}{D}\left(\nu + \frac{\theta}{2(1+\eta)}\right) \qquad (3.31)$$

$$c^*_{22} = \frac{E}{D}\left(1 + \frac{\theta}{1+\eta}\right)$$

$$G_3 = \frac{E}{2(1+\nu)} \; ; \quad c^*_{11}\, c^*_{22} - c^*_{12} = \frac{E^2}{D}$$

where $D = \left(1 + \frac{\theta}{(1+\eta)}\right)\left(1 + \frac{\theta}{2(1-\eta^2)}\right) - \left(-\nu - \frac{\theta}{2(1+\eta)}\right)^2$

All the relations (3.11) on the moduli are therefore satisfied. We also need

$$c^*_{12} + 2\,G_3 = \frac{E}{D}\left(\nu + \frac{\theta}{2(1+\eta)}\right) + \frac{E}{1+\nu} \qquad (3.32)$$

and it is again verified that (3.10) is satisfied, too.

3.5 CRITICAL STRESS AT BIFURCATION

The bifurcation stress formula appropriate to a corner on the Tresca yield surface is obtained by inserting (3.31) and (3.32) in (3.17); it can be written as

$$\frac{12\,\sigma b^2}{\pi^2\,t^2}\left(\frac{1-\nu^2}{E}\right) = \text{Min}\ F_T\left(m,\theta,\eta\,\frac{a}{b},\nu\right) \qquad (3.33)$$

where the function F_T of the indicated variables is

$$F_T = \frac{1-\nu^2}{D}\left[\left(\frac{mb}{a}\right)^2 \frac{2-2\eta^2+\theta}{2(1-\eta^2)} + 2\left\{\frac{2\nu+2\eta\nu+\theta}{2(1+\eta)} + \frac{D}{1+\nu}\right\} + \left(\frac{a}{mb}\right)^2\left(\frac{1+\eta+\theta}{1+\eta}\right)\right] \qquad (3.34)$$

in which D itself is a function of (θ, ν, η) as listed in (3.31). The formula appropriate to a smooth yield surface N = 1, essentially corresponding to the case of Mises yield surface, has been given by Sewell (1964); it can expressed as

$$\frac{12\,\sigma b^2}{\pi^2\,t^2}\left(\frac{1-\nu^2}{E}\right) = \text{Min}\left(m,\ \theta,\ \frac{a}{b},\ \nu\right) \qquad (3.35)$$

where the function F_M is

$$F_M = \frac{\left(\frac{mb}{a}\right)^2\left(1+\frac{\theta}{6}\right) + \left(\frac{a}{mb}\right)^2\left(1+\frac{2\theta}{3}\right)+2+\frac{\theta(1-2\nu)}{3(1+\nu)}}{\left[1+\left(\frac{5-4\nu}{6}\right)\theta\right]/\left(1-\nu^2\right)} \qquad (3.36)$$

The aim is now to find out whether the new formula (3.33), appropriate to a yield surface corner with hardening coupled according to the factor η , can predict a different bifurcation stress from that given by formula (3.35) corresponding to the Mises case. A numerical investigation has been carried out. Throughout we let

the value of the Poission's ratio $\nu = 1/3$. The right side of (3.33) and (3.35) were plotted in the range $0 \leq \theta \leq 10$ for ten aspect ratios in the range $a/b = 0.5, 1, 1.5 - - - 4.5, 5$. It was found adequate in (3.33) to fix on five values of η within (3.29), namely $\eta = -0.9, -0.6, 0, 0.5$ and 0.9. This makes six plots in all for each aspect ratio. Figs. 3.3 - 3.5 show such plots for $a/b = 1.0, 1.5, 3.0$. Also F_T and F_M were plotted as functions of a/b for a given value of θ and η (in case of F_T) . Fig. (3.6-3.8) show some sample plots, which are the results of minimisation of F_T and F_M with respect to positive integral values of m. The m value for an aspect ratio seems to depend on both θ and η . For example, for $a/b = 1.2$ the m is 1 for $\theta = 0$ (elastic case), it equals 2 when $\theta = 10$ for the Mises case as well as for the Tresca case ($\eta = 0.9$) but takes on the value 1 for the latter case if $\eta = -0.9$.

Next we minimise (3.33) and (3.35) with respect to continuous m. The associated formula for the Tresca case is

$$\frac{12\sigma b^2}{\pi^2 t^2}\left(\frac{1-\nu^2}{E}\right) = F_T^*(\theta, \eta, \nu) \qquad (3.37)$$

when $$F_T^* = \frac{2(1-\nu^2)}{D}\left[\left\{\frac{(2-2\eta^2+\theta)(1+\eta+\theta)}{2(1-\eta^2)(1+\eta)}\right\}^{1/2} + \frac{2\nu+2\eta\nu+\theta}{2(1+\eta)} + \frac{D}{1+\nu}\right] \qquad (3.38)$$

The minimisation can actually be carried out with respect to the parameter mb/a and as in (3.18), the minimum is attained at

$$\frac{mb}{a} = \left[\frac{1 + \theta/(1+\eta)}{1 + \theta/2(1-\eta^2)}\right]^{1/4} \tag{3.39}$$

The corresponding formula for the Mises case is

$$\frac{12\ \sigma b^2}{\pi^2\ t^2}\left(\frac{1-\nu^2}{E}\right) = F_M^* \quad (\theta, \nu) \tag{3.40}$$

where

$$F_M^* = \frac{\left[2\left(1+\frac{2\theta}{3}\right)\left(1+\frac{\theta}{6}\right)^{1/2} + 2 + \frac{\theta(7-2\nu)}{3(1+\nu)}\right]}{\left[1 + \left(\frac{5-4\nu}{6}\right)\theta\right] /(1-\nu^2)} \tag{3.41}$$

The analytical minimum is obtained, in this case, at the value

$$\frac{mb}{a} = \left[\frac{6 + 4\theta}{6 + \theta}\right]^{1/4}. \tag{3.42}$$

The functions F_M^* and F_T^* were plotted for many possible combinations of the variables θ and η. The horizontal lines in Fig. 3.6-3.8 are, in fact, these functions. When compared with the graphs of integer minima, it was found that there was not much difference between the use of integral or continuous m in the minimisation, at least for $a/b \geq 3$. This might have been expected from what is known about the classical long plate approximation in elasticity, but it was necessary to check in the present

context also. It is, therefore, adequate to concentrate attention upon six curves (five functions F_T^* and the function F_M^*) shown in Fig. 3.9 obtained by minimising the right side of (3.37) and (3.40) with respect to continuous m .

It is to be noted from Fig. 3.9 that all the curves for Tresca case lie below the Mises curve as θ increases from zero. Infact, an increase in θ from zero represents the fall-off in tangent-modulus from E, as will be shown later. Even for independent hardening (η = 0) for the two yield surface facets, the function F_T^* drops considerably below the Mises curve (F_M^*) and as η increases into positive values the gap widens at an increasing rate. As $\eta \rightarrow 1$, it is clear that a drametic reduction in the ratio F_T^* / F_M^* takes place even while θ is very small. We cannot actually put $\eta = 1$ as it will violate (3.29), but for $\eta = 0.9$ we find that $F_T^* / F_M^* \doteq 1/1.6$ is achieved by the time θ has reached about 3.0 . In the absence of clear information from physics about the value of η , we have investigated the effect of all $\eta^2 < 1$. It turns out, as seen in Fig. 3.9, that the above mentioned sensitivity of F_T^* to variations in positive value of η does not occur for negative η . More-over, the F_T^* curves for $\eta > 0$ do not intersect each other, in contrast to those for $\eta < 0$.

The minimising values of mb/a, given by (3.39) and (3.42), are shown in Fig. 3.10, as a function of θ. There is something special when the coupled hardening has a value $\eta = 0.5$; the matrix of moduli in (3.30) becomes isotropic instead of anisotropic, just as for an elastic solid. This is why the horizontal line at mb/a = 1 is obtained for $\eta = 0.5$ in Fig. 3.10. It implies the continuation into the plastic regime of the elastic property that for plates of integral ratio a/b, the buckling half-wavelength and the aspect ratio coincide and are independent of the amount of strain (represented by θ here) before bifurcation. Fig. 3.10 also shows that the critical value of mb/a falls with increasing θ when $0.5 < \eta < 1$ in constrast to its rising behaviour for $-1 < \eta < 0.5$ and in the Mises case.

3.6 NUMERICAL CALCULATION OF BIFURCATION STRESS

It has been shown in the last section that substantial differences do occur between the right sides of the bifurcation stress formulae (3.33), (3.35) and their respective lower bounds. As shown in Fig. 3.9, the error involved is negligible if $a/b \geq 3$. We now wish to compare the bifurcation stresses actually predicted in this range of a/b by the formulae (3.37) and (3.40), in situations corresponding to an actual test.

In order that any F_T^* , θ point or F_M^* , θ point in Fig. 3.9 can actually represent a bifurcation value of θ (or of the hardening parameter h if E is constant), the strain history must have been such that the corresponding bifurcation stress σ required by (3.37) or (3.40) be at a point of the stress-strain curve which is compactable with this value of θ (or of h).

To achieve this, and so to obtain and compare some numerical values of the bifurcation stress, we make some assumptions which will enable us to determine the left side of (3.37) and (3.40) as an increasing function of θ. Then these curves can be superimposed on the decreasing curves of Fig. 3.9; the corresponding intersections will determine actual bifurcation values of σ and θ.

Suppose that the instantaneous tangent modulus E_t in compression is adequately expressed as $1/c_{11}$ from (3.8), by inserting $D\tau_{22}/Dt = 0$ there to give $\varepsilon_{11}=c_{11}\, D\tau_{11}/Dt$. From (3.10) and (3.31) it follows that

$$\frac{E}{E_t} = 1 + \frac{\theta}{1+\eta} \qquad (3.43)$$

for the Tresca case. For the Mises case, it is given by (Sewell, 1964)

$$\frac{E}{E_t} = 1 + \frac{2\theta}{3} \qquad (3.44)$$

It is to be noted that when $\eta = 0.5$, the two tangent moduli are the same functions of θ. This is the same amount of coupled hardening which results in some simplifications mentioned in connection with Fig. 3.10.

It is clear from (3.43) or (3.44) that as progress is made from yield along a stress-strain curve with monotonically decreasing E_t, the parameter θ will monotonically increases from zero. We neglect any variation with θ which the width /thickness ratio (b/t) may have. Then, with any given stress-strain curve (and fixed ν), the left side of (3.37) and (3.40) is expressible as a decreasing function of E_t. For each fixed η it will be an increasing function of θ , from (3.43). For example, consider a two-segment stress-strain curve represented by the Ramberg-Osgood relation of the type

$$\frac{e}{e_y} = \frac{\sigma}{\sigma_y} + \alpha \left(\frac{\sigma}{\sigma_y}\right)^n \qquad (3.45)$$

where α, and n are material parameters, e_y is the yield strain which equals σ_y/E, σ_y is the yield stress and e is the compressive strain such that $de/d\sigma = 1/E_t$. Hence

$$\frac{E}{E_t} = 1 + \alpha n \left(\frac{\sigma}{\sigma_y}\right)^{n-1} \qquad (3.46)$$

and therefore from (3.43) we get

$$\sigma = \sigma_y \left[\frac{\theta}{(1+\eta)\alpha n}\right]^{\frac{1}{(n-1)}} \qquad (3.47)$$

Now, the left side of (3.37) becomes

$$\frac{12\,\sigma b^2}{\pi^2\,t^2}\left(\frac{1-\nu^2}{E}\right) = \left[\frac{\theta}{\alpha n(1+\eta)}\right]^{1/(n-1)} \sigma^* \qquad (3.48)$$

in which $\sigma^* = \frac{\sigma_y}{E}\left(\frac{b^2}{t^2}\right)\frac{12(1-\nu^2)}{\pi^2}$ (3.49)

For the purpose of numerical calculation we take $\alpha = 0.1$, $n = 8$ (Needleman, 1973) and consider variations in σ^* which is identical to varying the ratio of σ_y/E or the ratio b/t (or both) because ν is fixed. For the sake of illustration, three values of σ^* (i.e. 2.5, 3.0, 3.5)+ are taken. For each σ^* the rising curves (3.48) are obtained for five values of $\eta = -0.9, -0.6, 0, +0.5, +0.9$ and are plotted in Fig. (3.11-3.13). We recall from (3.44) that the curve (3.48) for $\eta = 0.5$ will also be for the Mises case. The curves from Fig. 3.9 are repeated in each of the Fig. 3.11-3.13; the corresponding intersections are bifurcation points. The common values of the two sides of (3.37) and (3.40) at bifurcation, as determined from these intersection curves, are shown in Table 3.1.

The conclusion to be drawn is that the occurance of a corner on the yield surface at the local stress point

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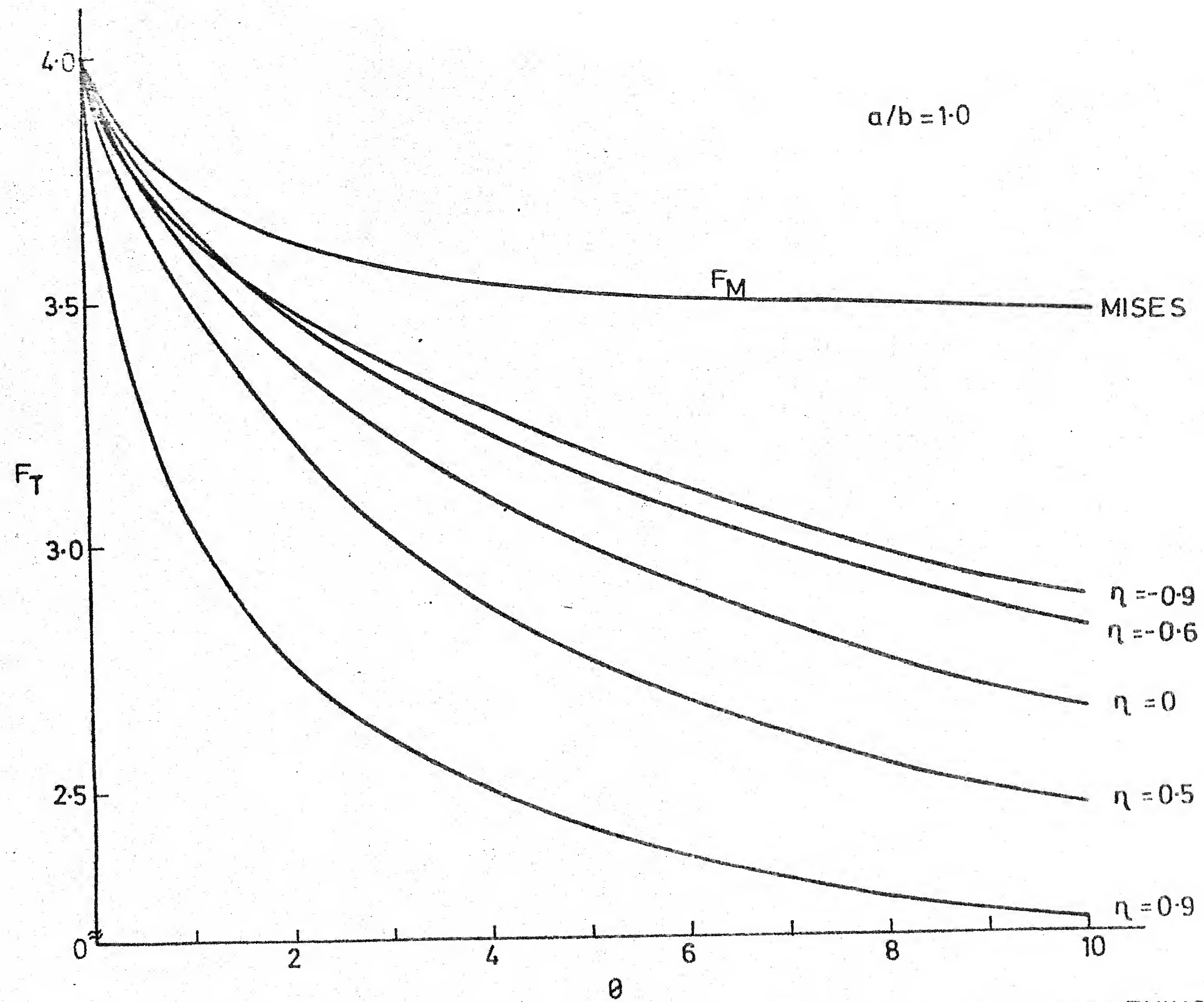
\+ go get an idea about the plate thickness, we note that for steel (σ_y= 2500 kg/cm^2, E=2.1 x 10^6 kg/cm^2) as σ^* varies from 2.5 to 3.5, b/t varies from 44 to 52.

can cause a lowering of the bifurcation stress when the coupled hardening coefficient does not assume a negative value. For $\eta \geq 0$, the reduction does exist. In particular, the lowering can be present even when hardening of the facets making up of the yield surface corner is uncoupled ($\eta = 0$), and a greater reduction in the bifurcation stress can occur if the coupling extends much beyond the value $\eta = 0.5$. It is to be noted that this reduction has been achieved even though the shear modulus retained its elastic value throught.

TABLE 3.1

COMPARISON OF THE BIFURCATION STRESS OBTAINED BY USING THE TRESCA AND THE MISES YIELD CRITERIA

σ^*	Mises Bifurcation Stress F_M^*	Tresca Bifurcation Stress F_T^*				
		$\eta = 0.9$	$\eta = 0.5$	$\eta = 0.0$	$\eta = -0.6$	$\eta = -0.9$
2.5	3.400	2.630	2.880	3.100	3.300	3.600
3.0	3.470	2.900	3.200	3.400	3.580	3.770
3.5	3.650	3.150	3.450	3.585	3.750	3.900



FIG·3·3 VARIATION IN CRITICAL STRESS FUNCTIONS F_T & F_M WITH HARDENING PARAMETER θ (FOR a/b=1·0)

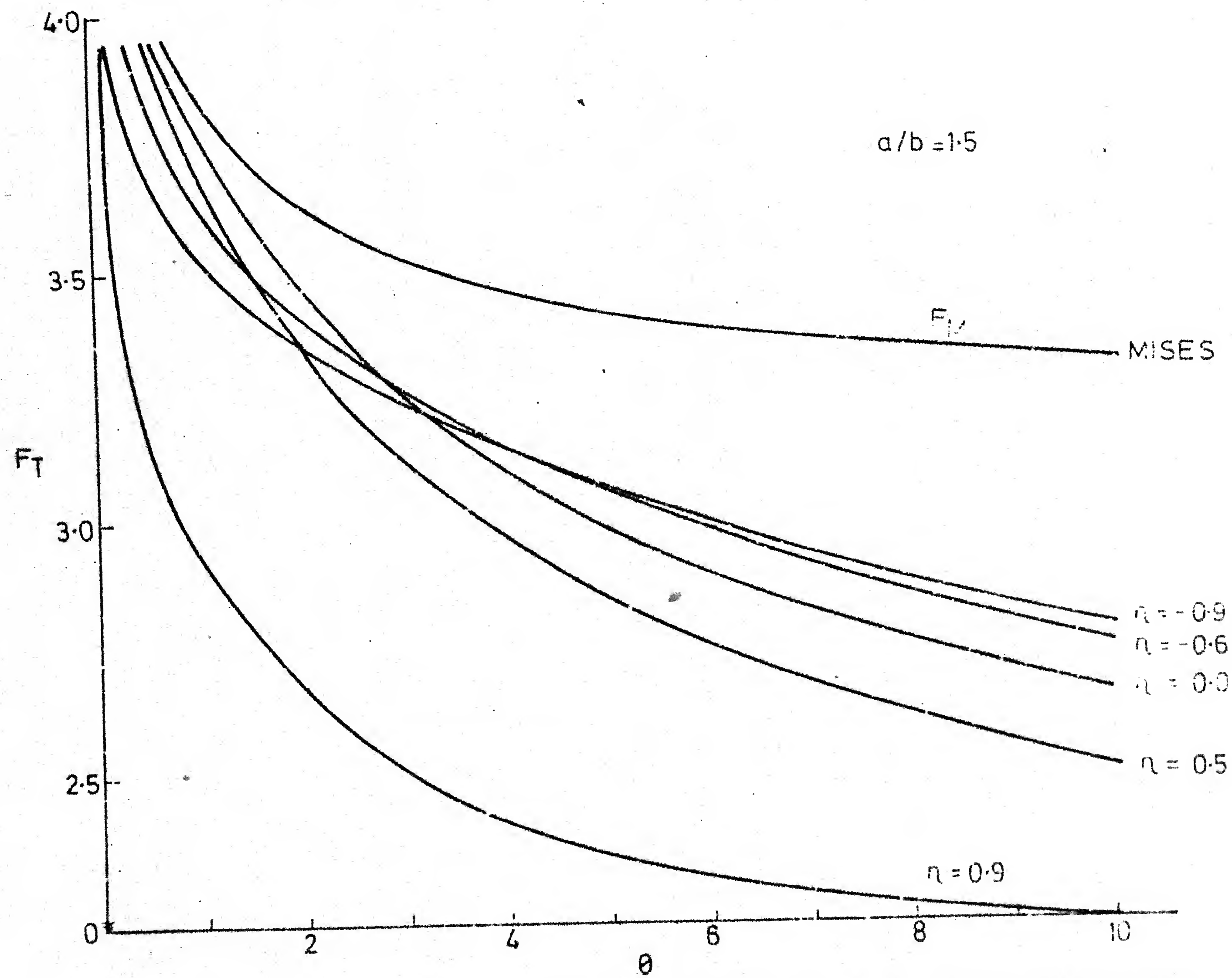


FIG.3·4 VARIATION IN CRITICAL STRESS FUNCTIONS F_T & F_M WITH HARDENING PARAMETER θ (FOR a/b=1·5)

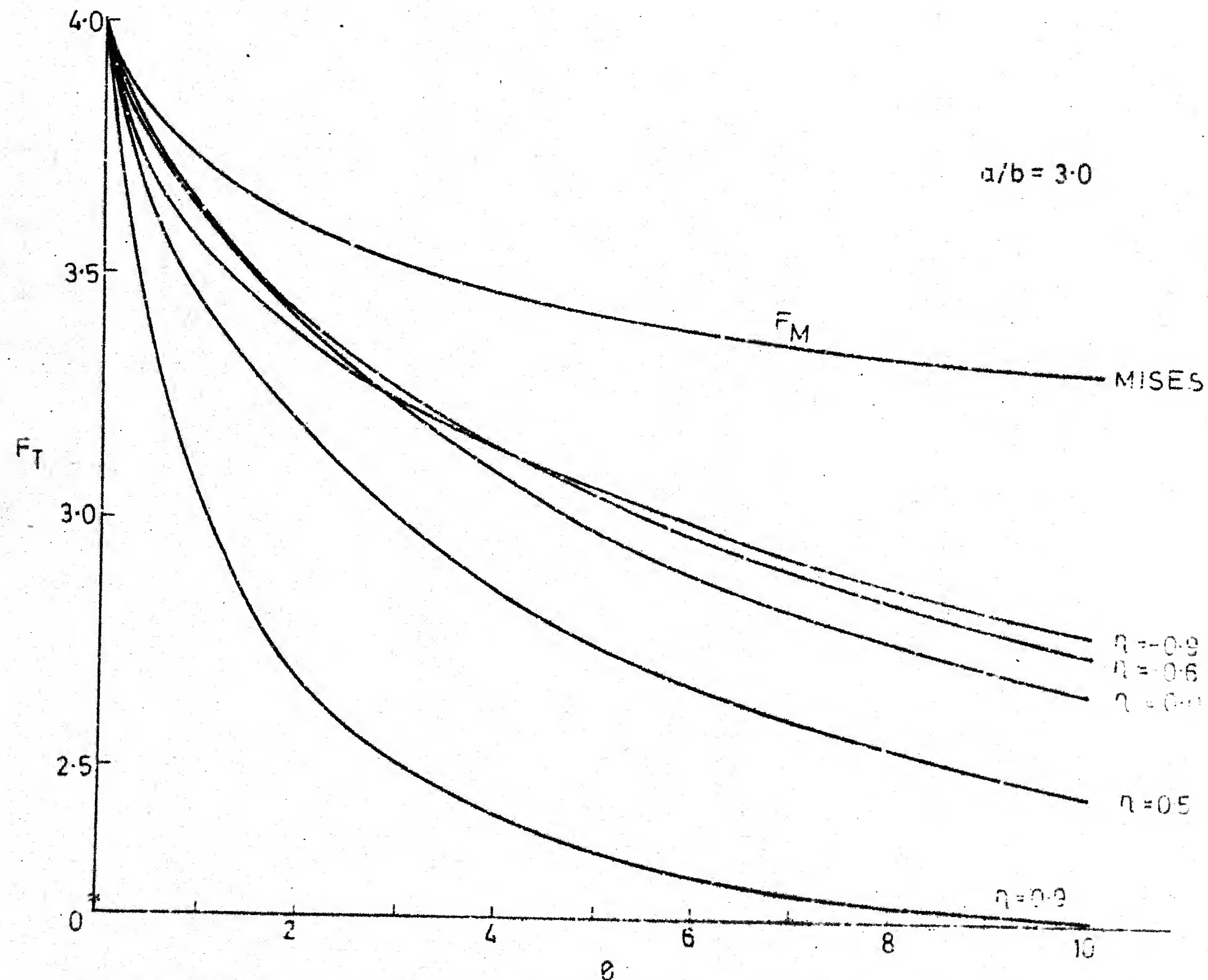


FIG. 3·5 VARIATION IN CRITICAL STRESS FUNCTIONS F_T & F_M WITH HARDENING PARAMETER θ (FOR a/b 3·0)

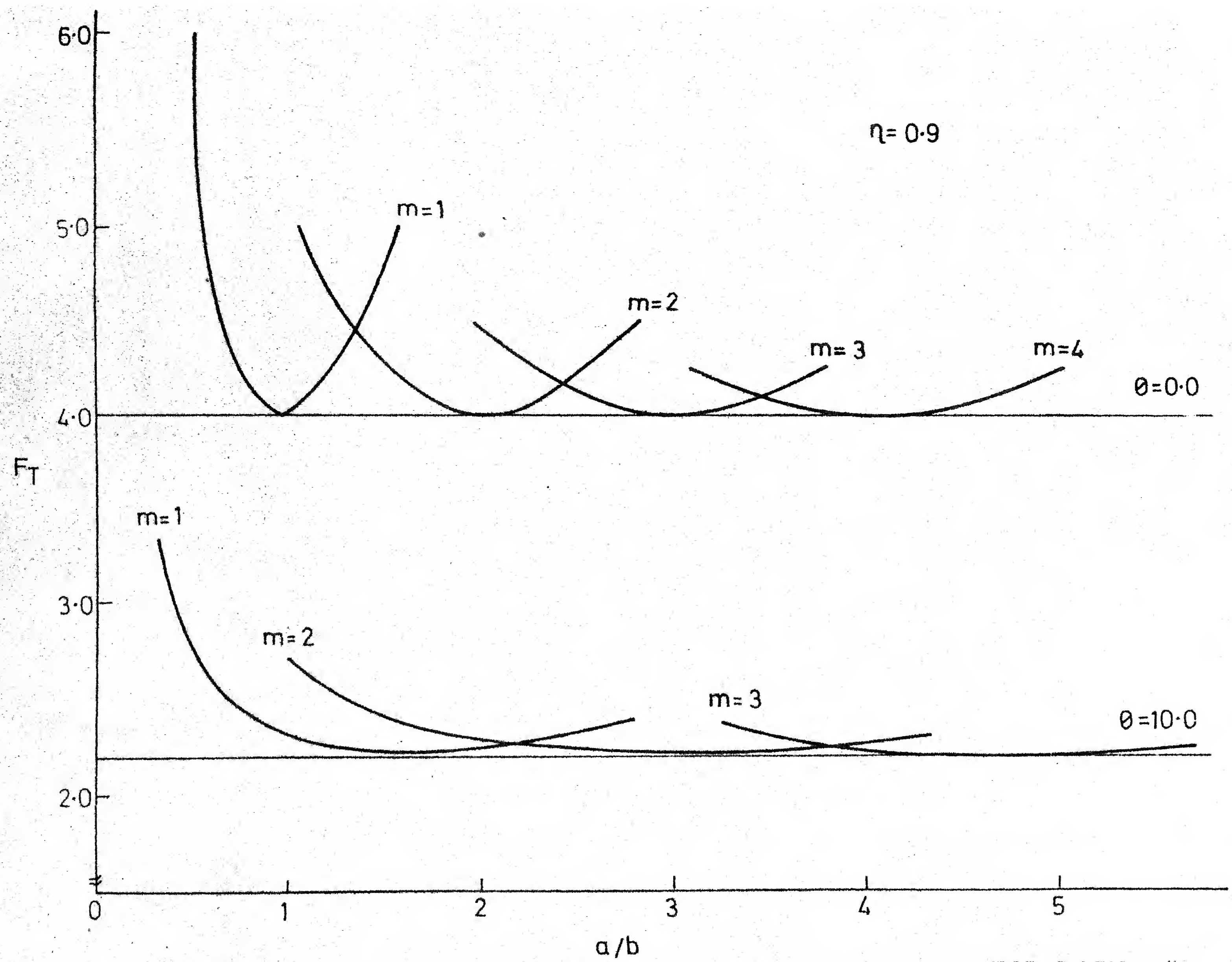


FIG.3·6 VARIATION IN CRITICAL STRESS FUNCTION F_T WITH ASPECT RATIO a/b (FOR $\eta = 0{\cdot}9$)

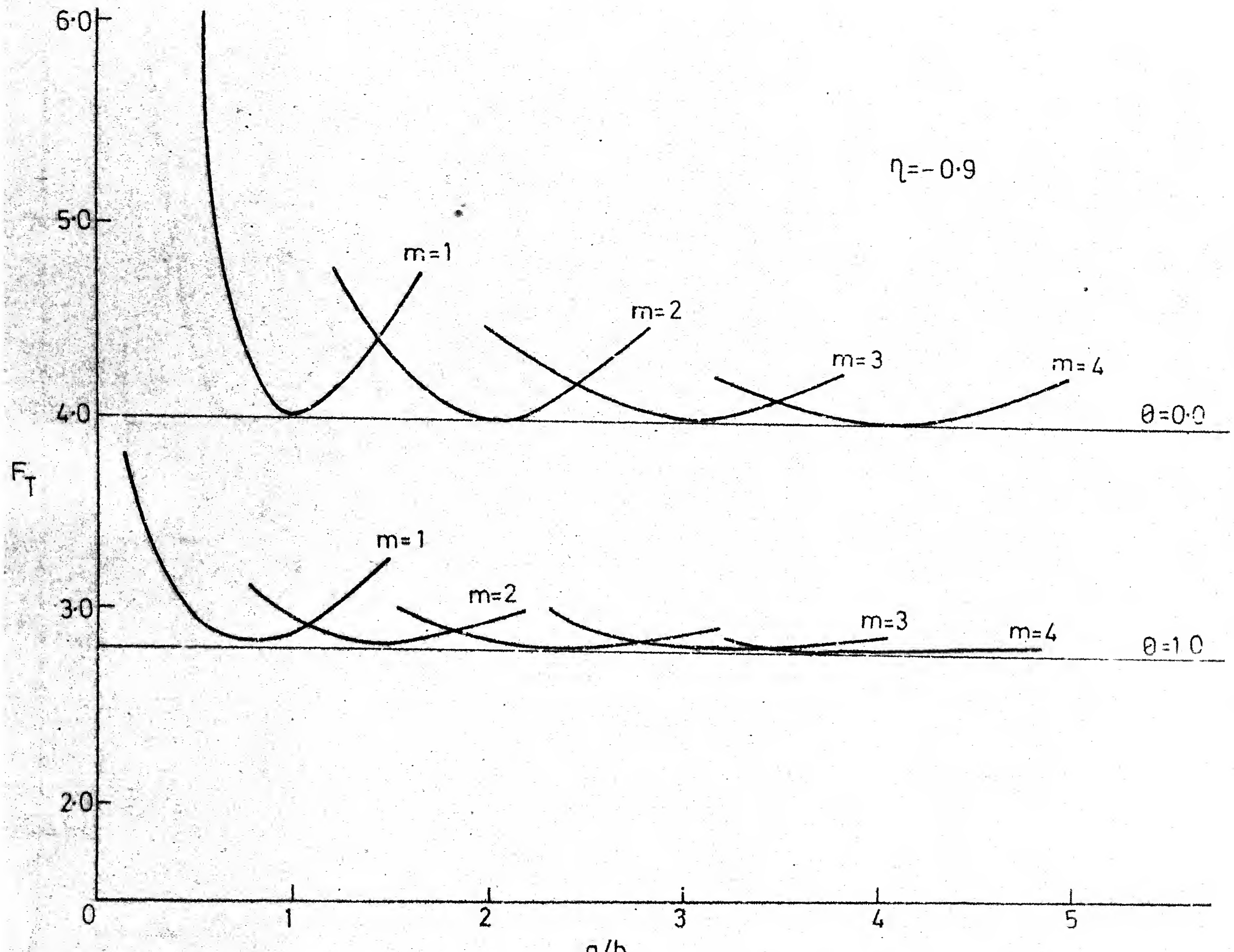


FIG. 3·7 VARIATION IN CRITICAL STRESS FUNCTION F_T WITH ASPECT RATIO a/b (FOR $\eta = -0{\cdot}9$)

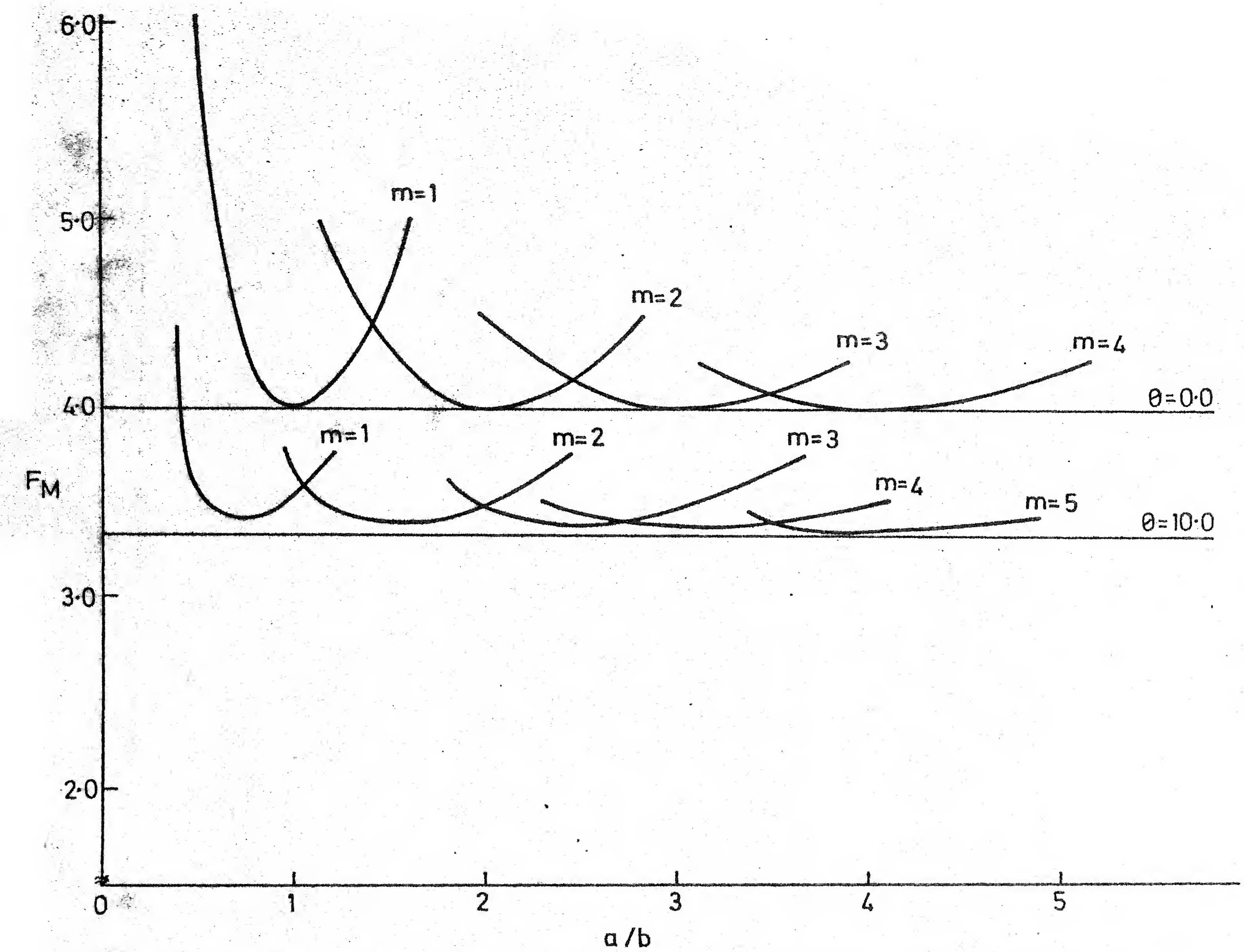


FIG.3·8 VARIATION IN CRITICAL STRESS FUNCTION F_M WITH ASPECT RATIO a/b

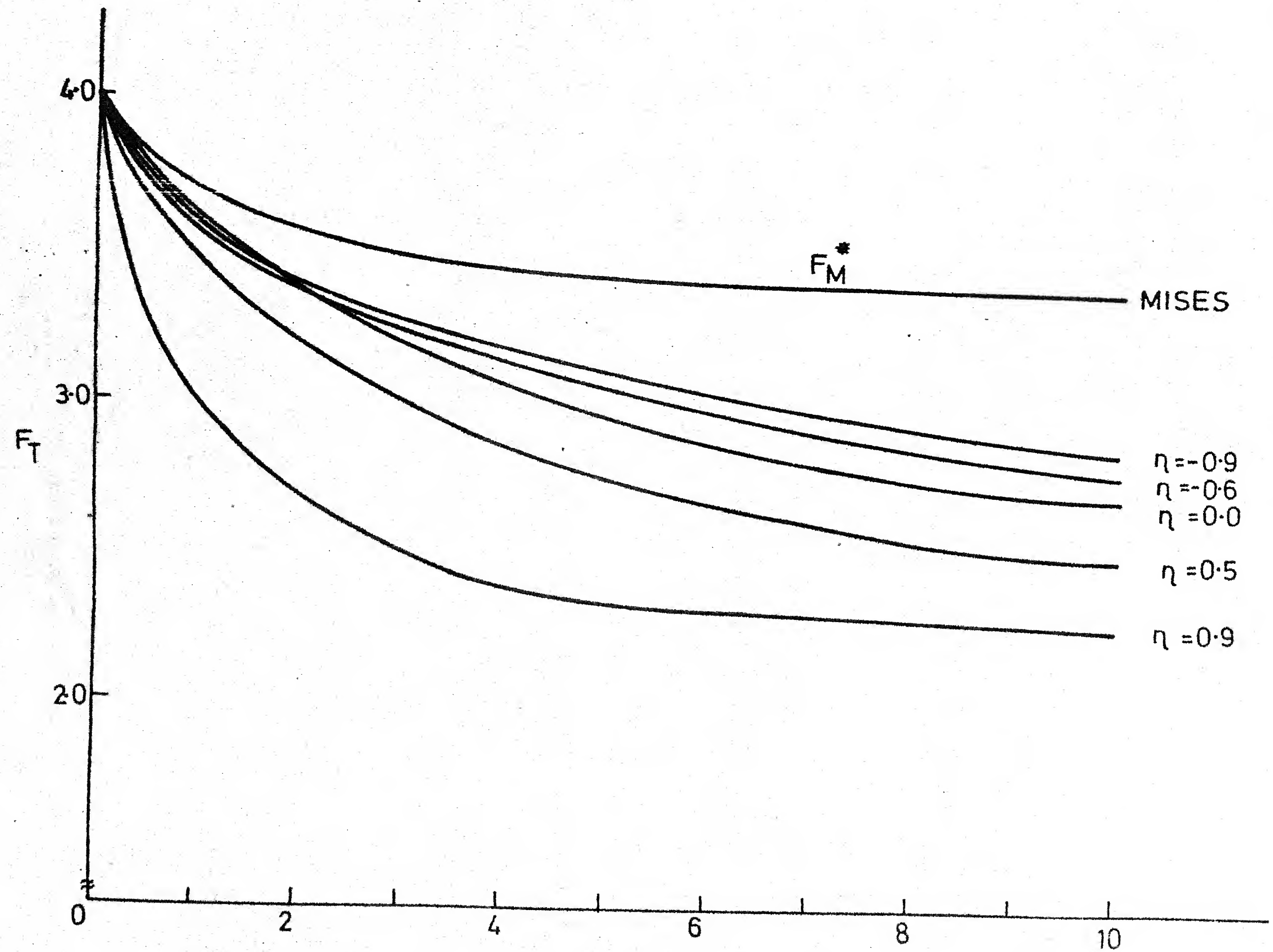


FIG.3.9 VARIATION IN CRITICAL STRESS FUNCTIONS F_T^* & F_M^* WITH HARDENING PARAMETER θ

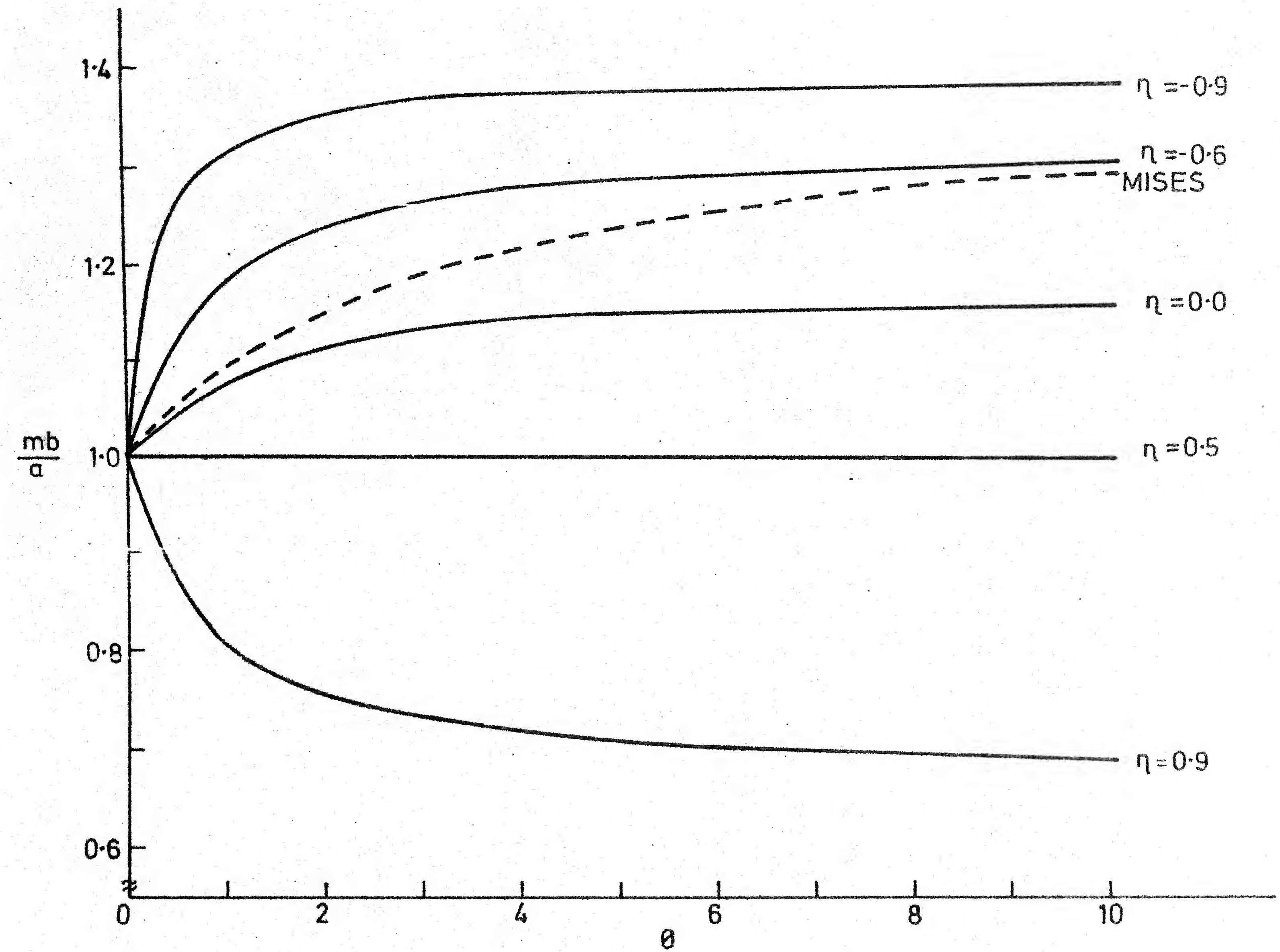


FIG.3·10 MINIMIZING VALUES OF mb/a AS A FUNCTION OF HARDENING PARAMETER θ

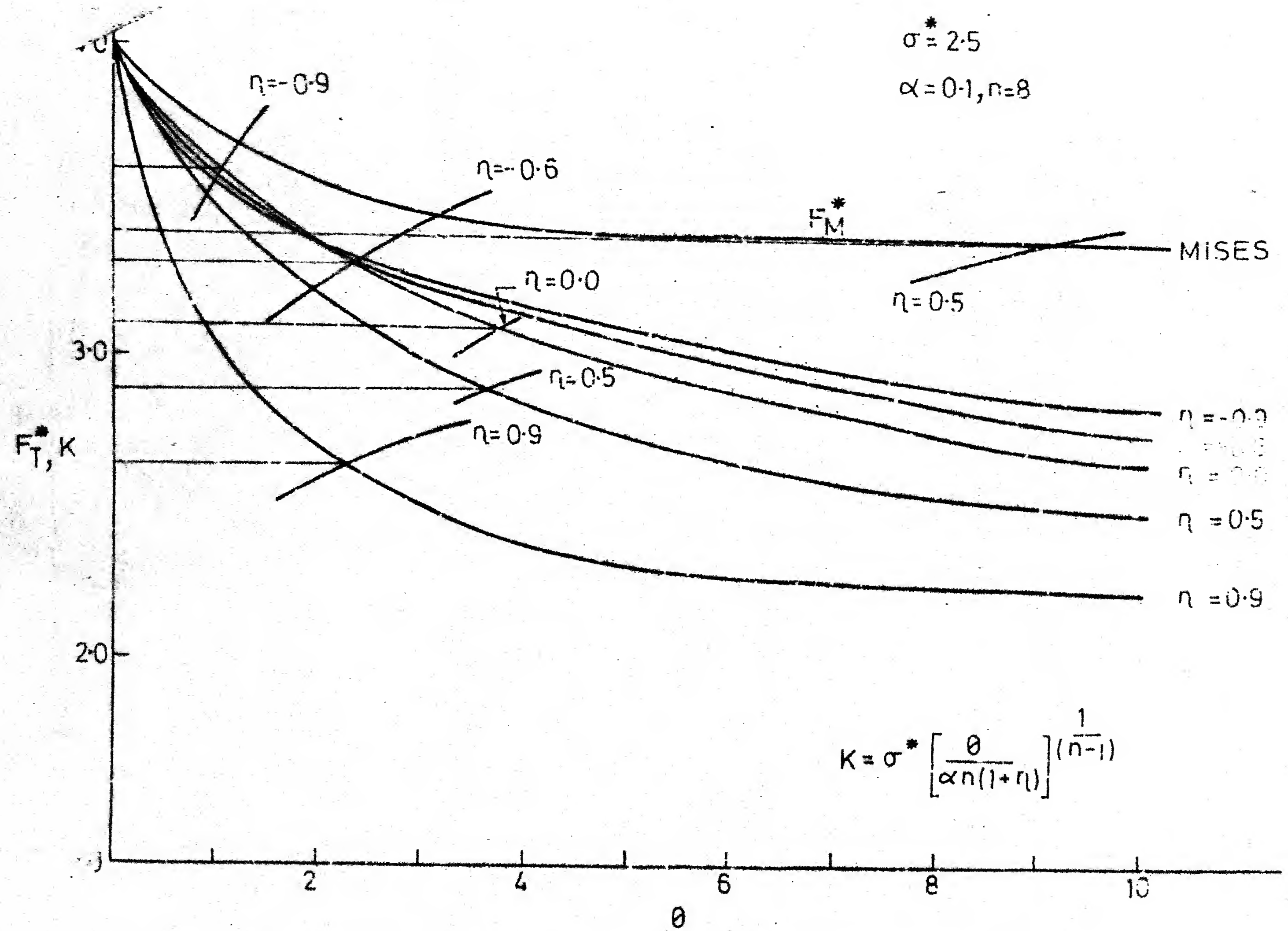


FIG.3·11 BIFURCATION STRESSES CORRESPONDING TO STRESS-STRAIN CURVE (EQ.3·45) FOR $\alpha = 0{\cdot}1$, $n = 8$, $\hat{\sigma} = 2{\cdot}5$

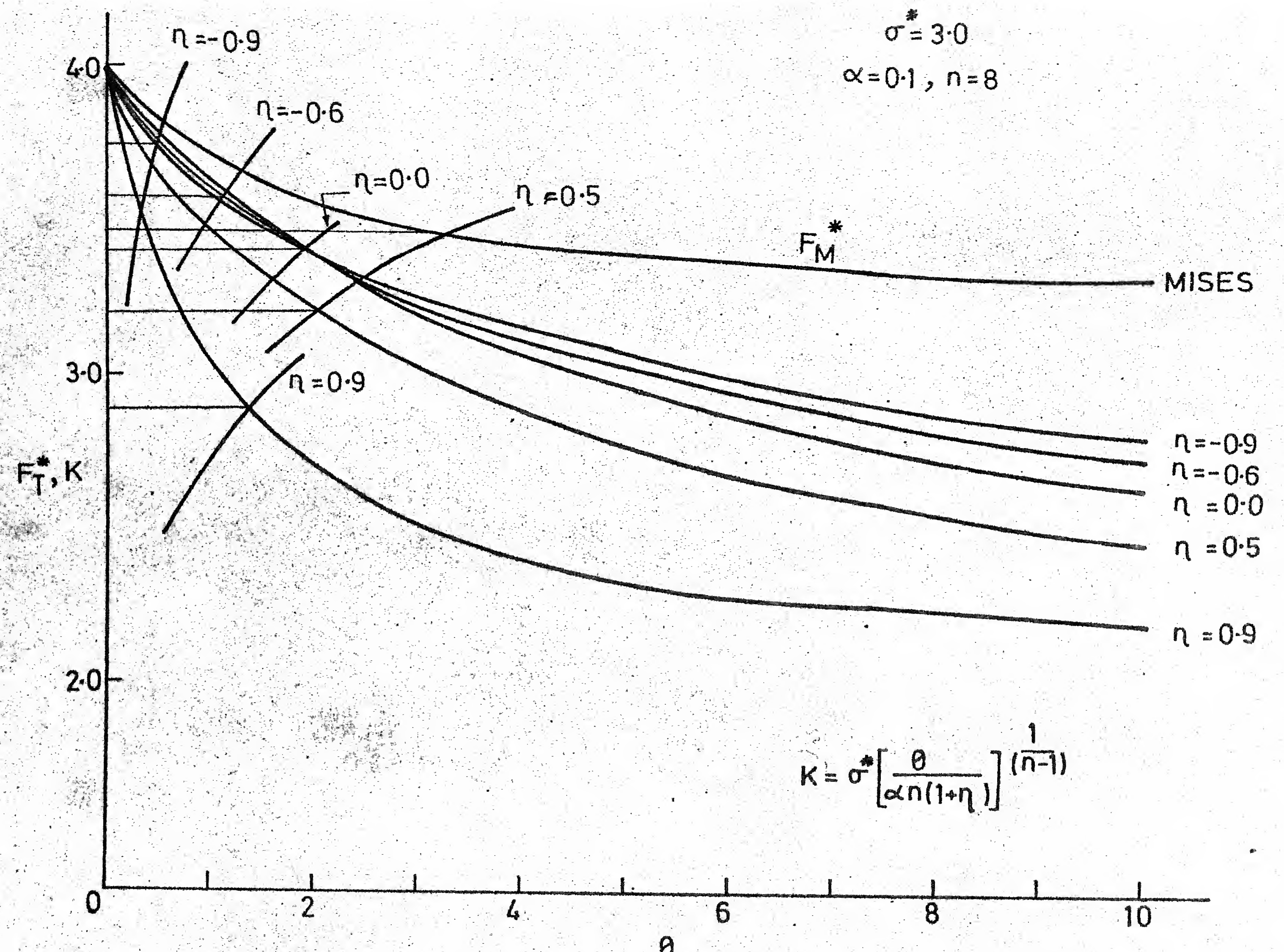


FIG. 3.12 BIFURCATION STRESSES CORRESPONDING TO STRESS-STRAIN CURVE (EQ. 3.45) FOR $\alpha = 0\cdot1$, $n = 8$, $\sigma^* = 3\cdot0$

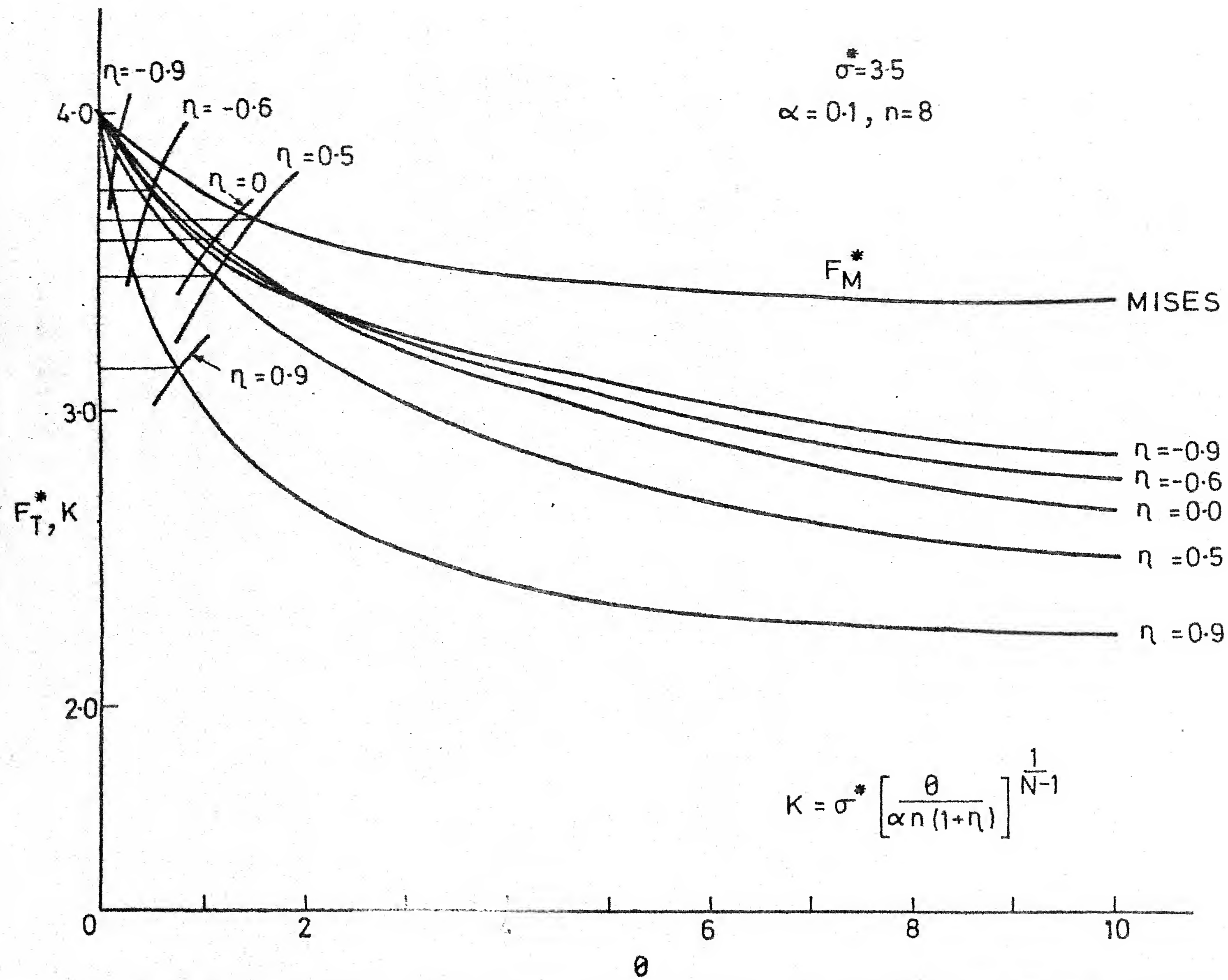


FIG.3·13 BIFURCATION· STRE[illegible] [illegible]RRESPO[illegible]NG TO STRESS-STRAIN CURVE (EQ.3·45) FOR $\alpha = 0{\cdot}1,\ n = 8,\ \sigma^* = 3{\cdot}5$

CHAPTER 4

ELASTIC PLATE BUCKLING

4.1 INTRODUCTION

The study of the buckling phenomenon in thin elastic rectangular plates has been carried out by many investigators in the light of small deformation theory. The aim of this chapter is to reexamine this problem under the present framework of uniqueness and stability, outlined in Chapter 2. The plate material is assumed isotropic in the ground state but is allowed to develope anisotropy during deformation; the material becomes orthotropic in the current state the stability of which is being investigated. The plate is assumed simply supported on all edges.

The method of analysis is similar to that in Chapter 3; the difference lies in the constitutive equations used. The plate is supposed to be made of a general incompressible elastic material by adopting the strain-energy function of Ogden (1972). The energy function contains a parameter α by varying which a large variety of solids can be generated. Incidently, $\alpha = 2$ will correspond to a neo-Hookean material. One of the advantages of using such functions is that the results

(i.e. the critical stress) can be expressed interms of moduli (namely the shear modulus in the present context) in the undeformed configuration. The rate constitutive behaviour of the material is taken in the form proposed by Hill (1970) which is available in a form simpler and more convenient than any other form involving strain-invariants.

4.2 CONSTITUTIVE RELATIONS

The principal Cauchy stresses σ_1, σ_2 , σ_3 may be regarded as functions of the three principal stretches a_1, a_2, a_3 measured from a ground state with respect to which the material is isotropic. The principal axes of the stresses are coincident with the axes of the Eulerian strain ellipsoid. The principal Kirchhoff stresses are defined simply by

$$\tau_1 = J\,\sigma_1 , \quad \tau_2 = J\,\sigma_2, \quad \tau_3 = J\,\sigma_3 \qquad (4.1)$$

where $J = a_1 a_2 a_3$. If the solid is hyperelastic with strain energy W per unit reference volume, we have

$$\tau_1 = a_1 \frac{\partial W}{\partial a_1} , \quad \tau_2 = a_2 \frac{\partial W}{\partial a_2} , \quad \tau_3 = a_3 \frac{\partial W}{\partial a_3} \qquad (4.2)$$

The rate of change of principal Kirchhoff stresses are given as

$$\dot{\tau}_1 = a_1 \frac{\partial \tau_1}{\partial a_1} e_{11} + a_2 \frac{\partial \tau_1}{\partial a_2} e_{22} + a_3 \frac{\partial \tau_1}{\partial a_3} e_{33}, \text{ etc.} \qquad (4.3)$$

where $\dot{a}_1/a_1 = e_{11}, \dot{a}_2/a_2 = e_{22}, \dot{a}_3/a_3 = e_{33}$ are the normal components of the Eulerian strain-rate e_{ij} on the axis of the Eulerian ellipsoid. Then, the incremental relations in terms of the Jaumann derivative can be expressed as (Hill, 1970)

$$\frac{D\tau_{11}}{Dt} = \frac{\partial}{\partial a_1}.(a_1\sigma_1)e_{11} + \frac{\partial}{\partial a_2}(a_2\sigma_1)e_{22} + \frac{\partial}{\partial a_3}(a_3\sigma_1)e_{33}, \text{etc.},$$

$$\begin{aligned} \frac{D\tau_{12}}{Dt} &= \frac{a_1^2 + a_2^2}{a_1^2 - a_2^2}(\sigma_1 - \sigma_2)\, e_{12}\ , \quad a_1 \neq a_2 \\ &= a_1 \frac{\partial}{\partial a_1}(\sigma_1 - \sigma_2)e_{12}\ , \quad (a_1 = a_2) \end{aligned} \quad \text{,etc.} \qquad (4.4)$$

which is in a form similar to (2.16) ideally suited for using in the uniqueness/stability criterion. The instantaneous moduli are then just the coefficients of the components e_{ij} in (4.4).

In case the solid is isotropic in the reference configuration, the potential W is a function of special choice of strain invariants (e.g. Green and Zerna, 1968) or of principal stretches a_i as considered by Ogden (1972). We consider the following strain-energy function for an incompressible elastic material

$$W = \mu\left(\frac{a_1^\alpha + a_2^\alpha + a_3^\alpha - 3}{\alpha}\right) \qquad (4.5)$$

where α is a real number and μ the mechanical parameter such that

$$\mu\alpha \quad > \quad 0 \tag{4.6}$$

is the shear modulus in the undeformed configuration. The principal Cauchy stresses for an incompressible solid are expressed as

$$\sigma_1 - \frac{1}{a_3 a_2}\frac{\partial W}{\partial a_1} = \sigma_2 - \frac{1}{a_1 a_3}\frac{\partial W}{\partial a_2} = \sigma_3 - \frac{1}{a_1 a_2}\frac{\partial W}{\partial a_3} = p \tag{4.7}$$

where p is an undetermined quantity. For the function W defined by (4.5) , the relations (4.7) become

$$\sigma_1 - \mu a_1^{\alpha} = \sigma_2 - \mu a_2^{\alpha} = \sigma_3 - \mu a_3^{\alpha} = p \tag{4.8}$$

such that

$$a_1 a_2 a_3 = J = 1 \tag{4.9}$$

which is the condition of incompressibility. Since the current state of the stress is the uniaxial compression along the x_1-direction (i.e., $\sigma_2 = \sigma_3 = 0$), we have from (4.9)

$$a_3 = a_2 = a_1^{-1/2} \tag{4.10}$$

so that we get

$$\frac{\sigma_1}{\mu\alpha} = \frac{a_1^{\alpha} - a_1^{-\alpha/2}}{\alpha} \tag{4.11}$$

4.3 APPLICATION OF UNIQUENESS CRITERION

The first term in the uniqueness/stability criterion (2.17) can be expanded in the following form in view of (4.4)

$$\frac{D\tau_{ij}}{Dt} \epsilon_{ij} = C_{ijkl}\, \epsilon_{ij}\, \epsilon_{ik} = \frac{1}{a_2 a_3} \frac{\partial}{\partial a_1}\left(a_1 \frac{\partial W}{\partial a_1}\right) \epsilon_{11}^2$$

$$+ \frac{1}{a_3 a_1} \frac{\partial}{\partial a_2}\left(a_2 \frac{\partial W}{\partial a_2}\right)\epsilon_{22}^2 + \frac{1}{a_1 a_2} \frac{\partial}{\partial a_3}\left(a_3 \frac{\partial W}{\partial a_3}\right) \epsilon_{33}^2$$

$$+ \frac{2}{a_1} \frac{\partial^2 W}{\partial a_2 \partial a_3}\epsilon_{22}\, \epsilon_{33} + \frac{2}{a_2} \frac{\partial^2 W}{\partial a_3 \partial a_1} \epsilon_{33}\, \epsilon_{11}$$

$$+ \frac{2}{a_3} \frac{\partial^2 W}{\partial a_1 \partial a_2} \epsilon_{11}\epsilon_{22} + \frac{2(a_1^2+a_3^2)}{(a_2^2 - a_3^2)}(\sigma_2 - \sigma_3)\epsilon_{23}^2$$

$$+ 2\frac{(a_3^2 + a_1^2)}{(a_3^2 - a_1^2)} (\sigma_3 - \sigma_1)\epsilon_{31}^2 + \frac{2(a_1^2+a_2^2)}{(a_1^2 - a_2^2)} (\sigma_1 - \sigma_2)\, \epsilon_{12}^2 \tag{4.12}$$

which after using (4.5) becomes

$$\frac{D\tau_{ij}}{Dt} = \mu\alpha\, a_1^{\alpha}\, \epsilon_{11}^2 + \mu\alpha\, a_2^{\alpha}\, \epsilon_{22}^2 + 2\left(\frac{a_1^2 + a_2^2}{a_1^2 - a_2^2}\right)(\sigma_1 - \sigma_2)\, \epsilon_{12}^2$$

$$+ 2\frac{(a_1^2 + a_3^2)}{(a_1^2 - a_3^2)} (\sigma_1 - \sigma_3)\, \epsilon_{13}^2 \tag{4.13}$$

where $a_1 a_2 a_3 = 1$ and $\sigma_1, \sigma_2, \sigma_3$ are given by (4.8). Also other terms in (2.17) get simplified to

$$- \sigma_1(\epsilon_{11}^2 + \epsilon_{12}^2 + \epsilon_{13}^2 - \omega_{12}^2 - \omega_{13}^2 + 2\epsilon_{12}\, \omega_{12} + 2\epsilon_{13}\, \omega_{13}) \tag{4.14}$$

so that (2.17) can now be written as

$$\int_V \Big[\frac{D\tau_{ij}}{Dt} e_{ij} - \sigma_1 \, (e_{11}^2 + e_{12}^2 + e_{13}^2 - \omega_{12}^2 - \omega_{13}^2 + 2e_{12}\,\omega_{12} + 2e_{13}\,\omega_{13}) \Big] dV > 0 \qquad (4.15)$$

where the first term is (4.13) itself.

In the current state, the plate has dimensions a x b x t. The plate is simply supported on edges $x_1 = 0$, a and $x_2 = 0$, b while the faces $x_3 = \pm\, t/2$ are free from applied traction - rates i.e.,

$$\dot{s}_{33} = \dot{s}_{31} = \dot{s}_{32} = 0 \quad \text{on } x_3 = \pm\, t/2 \qquad (4.16)$$

which are same as (3.6) . The boundary conditions can be taken care of in a manner similar to one in Chapter 3. The velocity field, infact is also identical to (3.12). In order to avoid the repetition of some unnecessary steps, we consider the following subset of the velocity field.

$$\begin{aligned} v_1 &= u - x_3 \frac{\partial w}{\partial x_1} \equiv u - x_3\, w_{,1} \\ v_2 &= u - x_3 \frac{\partial w}{\partial x_2} \equiv u - x_3\, w_{,2} \qquad (4.17) \\ v_3 &= w \end{aligned}$$

where u, v, w are the middle surface displacements and are functions of x_1, x_2 only. Again in the minimisation process of (4.15), it turns out that the best choice of the velocity functions u and v is $u \equiv v \equiv 0$.

Then the condition (4.15), for buckling not to set in, can be very much simplified to yield

$$\frac{t^3}{12}\int_0^a\int_0^b [\,\alpha a_1^{\alpha}\, w_{,11}^2 + \alpha a_1^{-\alpha/2}\, w_{,22}^2 + 2(a_1^{\alpha} - a_1^{-\alpha/2})\,\left(\frac{a_1^3+1}{a_1^3-1}\right) w_{,12}^2$$

$$+ (a_1^{\alpha} - a_1^{-\alpha/2})\,\left(\frac{12}{t^2}\, w_{,1}^2 - w_{,11}^2 - w_{,12}^2\right)]dx_1\, dx_2 > 0 \quad (4.18)$$

Since the plate is assumed simply supported on all edges, we take, as in (3.14)

$$w = \sin\frac{m\pi x_1}{a}\,\sin\frac{m\pi x_2}{b} \qquad (4.19)$$

when m and n are integers. The use of (4.19) reduces the condition (4.18) to requiring

$$m^4\,[\,\alpha a_1^{\alpha} - (a_1^{\alpha} - a_1^{-\alpha/2})\,] + \left(\frac{a}{b}\right)^4 n^4\,\alpha\, a_1^{-\alpha/2} + \left(\frac{a}{b}\right)^2 m^2 n^2$$

$$\left(\frac{a_1^3+3}{a_1^3-1}\right)\left(\frac{a_1^{\alpha}-a_1^{-\alpha/2}}{3}\right) + 3\,\frac{m^2a^2}{\pi^2t^2}\,(a_1^{\alpha}-a_1^{-\alpha/2}) > 0 \quad (4.20)$$

For a given a/b, t/a and α, the critical stretch a_1 is obtained by minimising (4.20) with respect to integers m and n. As expected the minimising value of n is always 1; the modified form of (4.20) is then

$$m^4[\,\alpha a_1^{\alpha} - (a_1^{\alpha}-a_1^{-\alpha/2})] + \left(\frac{a}{b}\right)^4 \alpha a_1^{-\alpha/2} + \left(\frac{a}{b}\right)^2 m^2\left(\frac{a_1^{\alpha}-a_1^{-\alpha/2}}{3}\right)$$

$$+3\left[\frac{a^2m^2}{\pi^2t^2}\,(a_1^{\alpha} - a_1^{-\alpha/2})\right] > 0 \qquad (4.21)$$

4.4 DISCUSSIONS

Once the plate material is specified (through parameters μ and α in (4.3)) and the plate dimensions are given (the aspect ratio a/b and the ratio t/a), the expression (4.21) can be minimised with respect to m to yield the critical value of the stretch a_1 at bifurcation. Having known the critical stretch, the critical stress can be calculated using(4.11). It may be mentioned here that the negative values of α are also admitted because μ can be negative too to make the shear modulus $\mu\alpha$ positive.

Six values of α were taken: α = -2.0, -1.5, -0.5, 0.5, 1.5, 2.0. The plate thickness/width ratio was maintained as 1/30 or 1/100. For each set, the aspect ratio was varied from 0.4 to 5.0 and in each case (4.21) was minimised with respect to the integral m to yield the critical stretch (and hence stress) . Fig. 4.1 and Fig.4.2 represent the variation in the buckling stress with aspect ratio for t/a = 1/30, 1/100 respectively, when α = 2.0 (i.e., neo-Hookean material). For other values of α, it is seen that the nondimensionalised critical stress increases as α decreases from 2.0 (excluding the value α = 1.0). The increase is, however marginal for t/a = 1/30 as can be seen from Table 4.1; and is almost negligible for t/a = 1/100.

TABLE 4.1

VARIATION OF CRITICAL STRESS WITH α FOR $t/a = 1/30$

	Critical Stress $\sigma / \mu\alpha \times 10^{-6}$			
a/b α	2.0	1.5	0.5	-1.5
0.4	7680	7696	7727	7791
1.0	3660	3661	3663	3668
1.6	3844	3846	3850	3859
2.4	3782	3785	3786	3789
3.6	3700	3701	3702	3711
5.0	3600	3661	3663	3668

It is interesting to note that though a large number of solids can be generated by varying α, the critical stress is not really sensitive to the variation in α. This is certainly true for the range of t/a considered here. No conclusion can be drawn for thicker plates because the analysis itself would require modification.

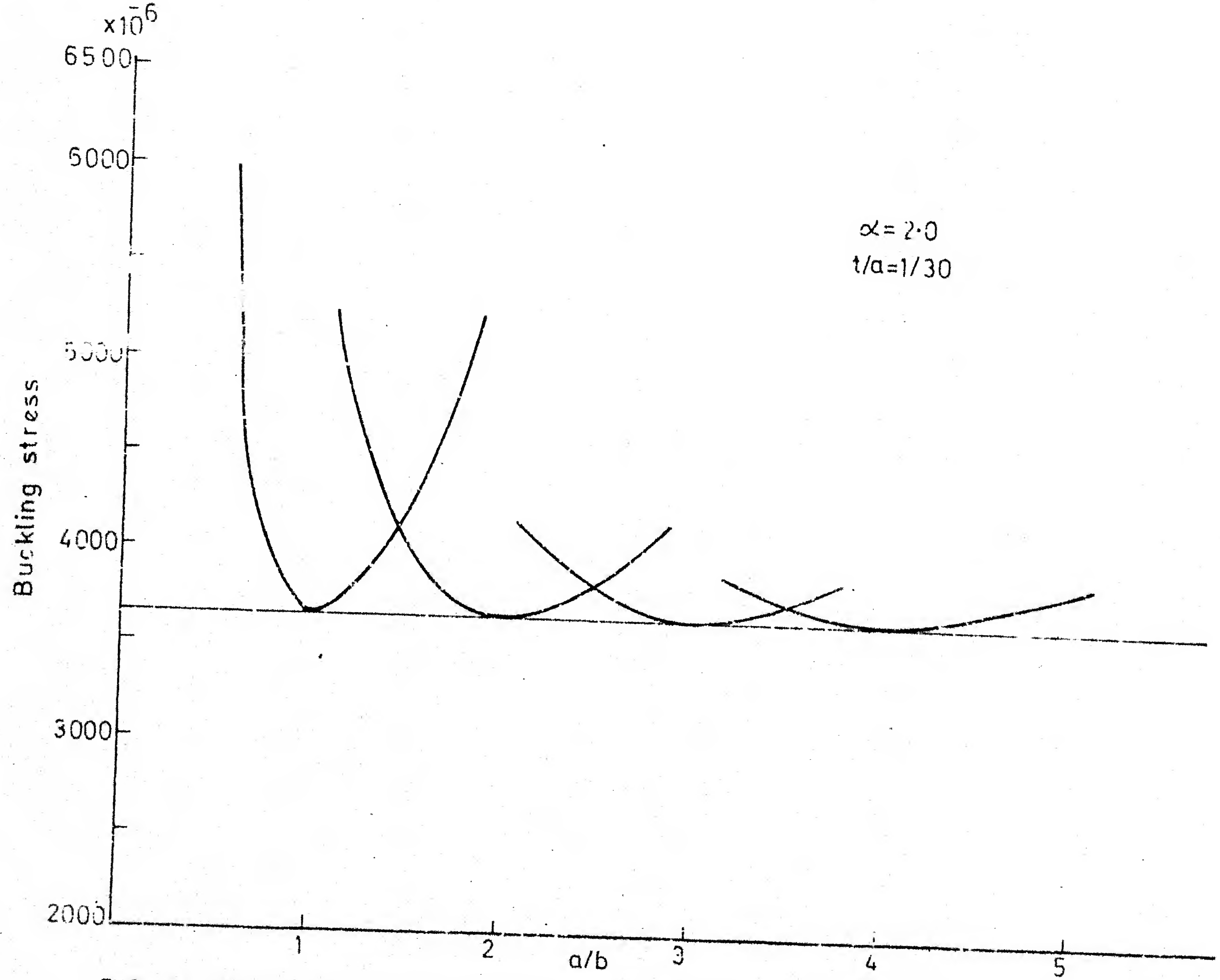


FIG.41 VARIATION IN BUCKLING STRESS WITH ASPECT RATIO (t/a =1/30)

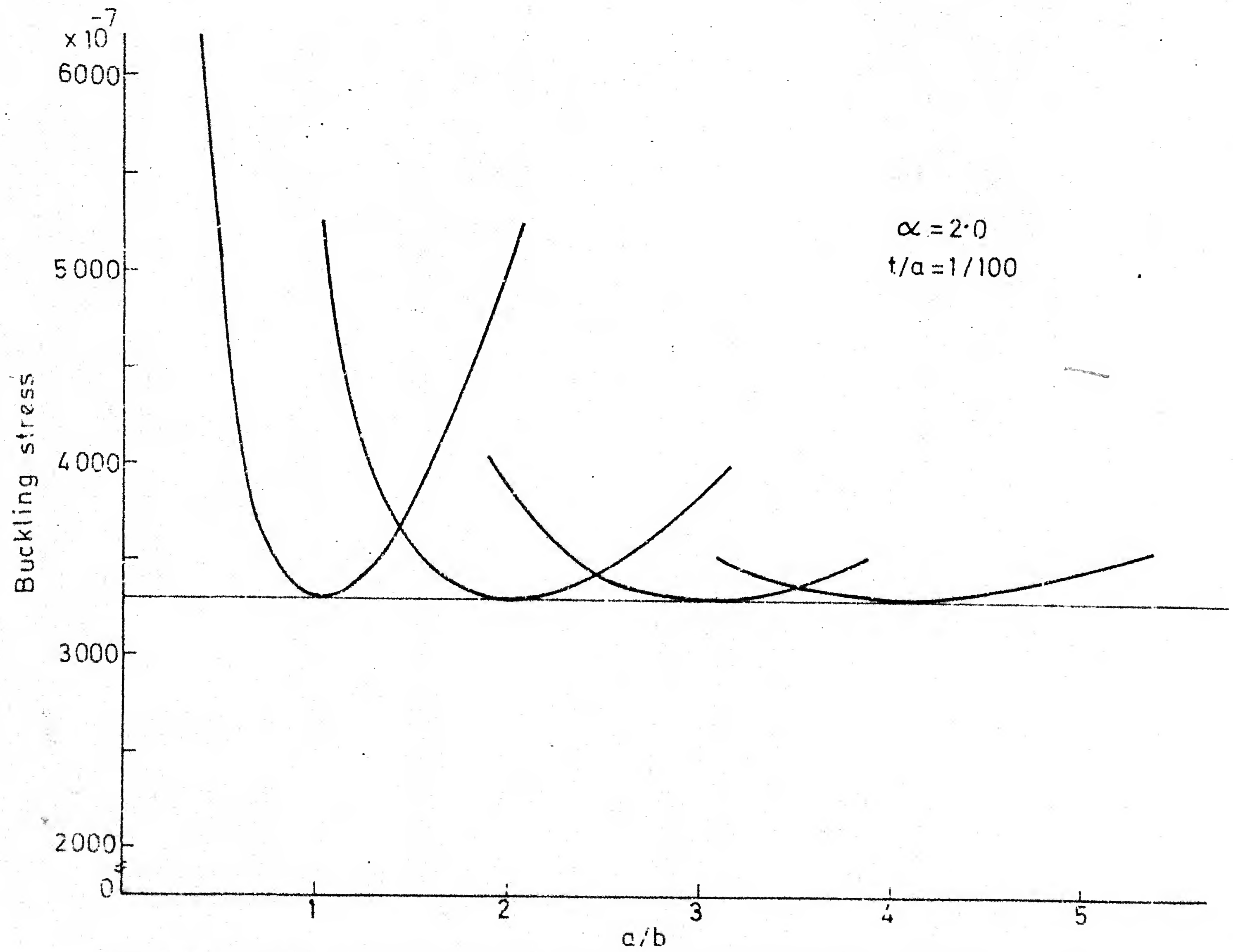


FIG.4.2 VARIATION IN BUCKLING STRESS WITH ASPECT RATIO (t/a=1/100)

CHAPTER 5

CONCLUSIONS

This thesis has been concerned with the investigation of the buckling phenomenon in thin elastic-plastic and elastic plates under finite axial compression. An attempt has been made to explain the so-called paradox in plastic plate buckling theory; the paradox is that the experimental buckling loads have been markedly lower than the theoretical bifurcation load obtained from the Prandtl-Reuss theory using the Mises smooth yield surface.

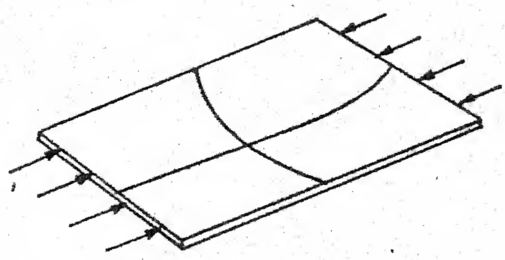
The bifurcation (buckling) stress has been calculated for a simply supported elastic-plastic plate, in the case when the stress is at a vertex of the Tresca yield surface. The effect of the coupled hardening between yield surface facets which meet at the vertex has been included. Fig.3.9 shows how the bifurcation stress depends on the tangent modulus of the compressive stress-strain curve. ______ the parameter θ is the reciprocal of the ordinary hardening normalized by Young's modulus E, so that as θ increases from zero the tangent modulus decreases from E. Fig. 3.9 also shows how the critical stress depends on the measure η of coupled hardening. This dependence has been found to become more sensitive as η increases from 0 to 1 .

Fig. 3.11 to Fig. 3.13 show how the Tresca bifurcation stress falls below that of Mises for some particular value of various parameter in the stress-strain curve which is of the Ramberg-Osgood type. It has been concluded that the reductions of order 10 to 30 percent may not be unrealistic when Mises is replaced by Tresca. Two separate new effects, infact, contribute to the reduction obtained. One is the corner itself when there is no coupling ($\eta = 0$); the other is coupling. Much of the previous literature is concerned with theories which embody a shear modulus which drops below its elastic value. The present investigation, however, has not taken this feature into account. The shear modulus does appear in the bifurcation stress formulae, but it is not the only parameter there, and the stated reduction have been obtained by using the above two effects together with an elastic shear modulus.

For the inelastic plate, simple probabilistic formulation has been given to predict the buckling stress when the random variables follow the normal distribution. For a desired reliability (or the probability of failure) and an assumed variation in material and geometric parameters, the buckling stress has been calculated and compared with those obtained in Chapter 3; the results based on the reliability analysis are always on lower side.

All the previous investigations concerning elastic plate buckling have presented results in terms of 'current' moduli of the material; these moduli, in turn, are related to the moduli in the undeformed state of the body and these relations are obtained by solving the problem of finite deformation from the initial state to the ground state. This has been obviated in the present investigation by considering the Ogden's strain-energy function; the critical stress, thus obtained is in terms of ground state shear modulus. The material is initially isotropic but is allowed to develop anisotropy through deformation. Although, only incompressible materials are considered, the analysis can easily be extended to utilise the strain energy function for compressible materials.

(a) m=1



(b) m= 2

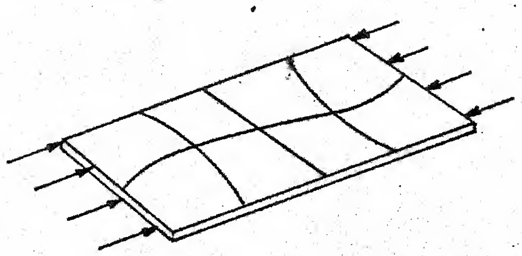


FIG. 5·1 BUCKLED FORM OF PLATE SUBJECTED TO IN-PLANE COMPRESSIVE LOADING

APPENDIX A

A RELIABILITY APPROACH TO INELASTIC BUCKLING

Consider a structure (or structural component) whose resistance is a random variable X , subjected to a random load Y with known density functions of $f_X(x)$ and $f_Y(y)$, respectively. The failure is then defined as the occurance of the event ($X \leq Y$) and the associated probability of failure is (Fig. A-1).

$$P_f = P\,(X \leq Y)$$

$$= \int_0^{\infty} \left[\int_0^{Y} f_X(x)\,dx \right] f_Y(y)\,dy$$

$$= \int_0^{\infty} F_X(y)\, f_Y(y)\,dy \qquad (A.1)$$

in which $F_X(\ )$ is the cumulative distribution function of X. When the load is $Y = y$ the probability of failure is $F_X(y)$. If X and Y are normal variates with means $\bar{X}$ and $\bar{Y}$ and standard deviations S_X and S_Y, the failure may be defined as ($X-Y < 0$) and the calculation of P_f is simplified considerably. In this case (X-Y) is also a normal variate with mean ($\bar{X} - \bar{Y}$) and variance ($S_X^2 + S_Y^2$). Then the probability of failure is given by

$$P_f = P\,(X - Y < 0)$$

and the equation (A.1) becomes

$$P_f = 1 - \phi\left[\frac{\bar{X} - \bar{Y}}{(S_X^2 + S_Y^2)^{1/2}}\right]$$

in which $\phi(z)$ is the tabulated cumulative probability of the standard normal variate (Z) for $Z \leq z$, where z is given by

$$z = \frac{\bar{X} - \bar{Y}}{(S_X^2 + S_Y^2)^{1/2}} \qquad (A.2)$$

In (A.2), z defines the quantum of probability of failure. Once the probability of failure is known, the reliability is given by

$$P_r = 1 - P_f$$

Now we apply the reliability theory to the problem of inelastic plate buckling in Chapter 3. We have the strength-stress relation for the inelastic plate given by (3.33) as

$$\left(\frac{1-\nu^2}{D}\right)\left[\left(\frac{mb}{a}\right)^2\left(\frac{2-2\eta^2+\theta}{2(1-\eta^2)}\right) + 2\left(\frac{2+2\eta\nu+\theta}{2(1+\eta)} + \frac{D}{1+\nu}\right)\right.$$

$$\left. + \left(\frac{a}{mb}\right)^2\left(\frac{1+\eta+\theta}{1+\eta}\right)\right] > \frac{12b^2}{\pi^2 t^2}\left(\frac{1-\nu^2}{E}\right)$$

From the above equation we observe that the expression on the left, henceforth called by K_c, denotes the resistance of the plate that has to be overcome if the buckling has to set in. The parameters θ, b, a have a random variation with means $\bar{\theta}$, $\bar{b}$, $\bar{a}$ and the variations

V_θ, V_b, V_a respectively.

We denote the expression on the right as K_s, since it is the actual stress that will develope at the time of buckling and is given by

$$K_s = \sigma \frac{12b^2}{\pi^2 t^2}\left(\frac{1-\nu^2}{E}\right) = \left[\frac{\theta}{\alpha n(1+\eta)}\right]^{\frac{1}{n-1}} \frac{\sigma_y}{E} \frac{12b^2(1-\nu^2)}{\pi^2 t^2}$$

$$= \left[\frac{\theta}{\alpha n(1+\eta)}\right]^{\frac{1}{n-1}} \cdot \sigma^*$$

involving σ_y, E as random variables. Let $\bar{\sigma}_y$, $\bar{E}$ be the mean values V_{σ_y}, V_E be the variations, respectively. So the reliability against buckling is given as

$$z = \frac{\bar{K}_c - \bar{K}_s}{\sqrt{S_{kc}^2 + S_{ks}^2}} \qquad (A.3)$$

where $\bar{K}_c$ is the mean expected strengths and $\bar{K}_s$ is the mean developed stress at buckling, and S_{kc}, S_{ks} are the respective standard deviations. For a given value of P_f the value of z can be obtained from the standard probability tables (Appendix B) and vice versa.

Since each of the variables in K_c and K_s is associated with a statistical distribution (assumed normal here), the over all distributions of K_c and K_s can be obtained by knowing the coefficients of variations of each of the variables. An approximate expression for

standard deviation (and hence variation) is given by

$$S_{kc}^2 = \sum_{i=1}^{n} \frac{\partial K_c}{\partial x_i} (x_1, x_2 \ldots\ldots x_n)$$

$$S_{ks}^2 = \sum_{i=1}^{m} \frac{\partial K_s}{\partial x_i} (y_1, y_2 \ldots\ldots y_m)$$

where $K_c = f\,(x_1, x_2, \ldots.x_n)$ and $K_s = g(\,y_1, y_2 \ldots.y_m)$. Using the above formulae the following relations can be derived:

$$V_{ks}^2 = V_{\sigma*}^2 + \frac{1}{(n-1)^2} V_\theta^2$$

where

$$V_{\sigma*}^2 = 4V_b^2 + V_E^2 + 4V_h^2 + V_{\sigma y}^2$$

$$V_\theta^2 = V_E^2 + V_h^2$$

Also

$$V_{kc}^2 = \left(\frac{1-\nu^2}{DK_c}\right)^2 \{ [\lambda^2 \left(\frac{2-2\eta^2+\theta}{2(1-\eta^2)}\right) + \left(\frac{1}{\lambda}\right)^2 \left(\frac{2+\eta+\theta}{1+\eta}\right) + \left(\frac{2\nu+2\eta\nu+\theta}{1+\eta}\right)]^2 V_D^2 + [\lambda^2 \left(\frac{1}{2(1-\lambda^2)}\right) + \frac{1}{1+\eta} + \frac{1}{\lambda^2(1+\eta)}]^2 \theta^2 V_\theta^2 + [\lambda^2 \left(\frac{2-2\eta^2+\theta}{1-\eta^2}\right) - 2\lambda^2 \left(\frac{1+\eta+\theta}{1+\eta}\right)]^2 V_\lambda^2 \}$$

where we have taken $\lambda = \frac{mb}{a}$. As already stated for a minimum with respect to continuous m, we have from Chapter 3,

$$\lambda = \left[\frac{2(1-\eta)(1+\eta+\theta)}{2-2\eta^2+\theta}\right]^{1/4}$$

so that $$V_\lambda^2 = \frac{\theta}{4\lambda^2}\left[\frac{(1-\eta^2)^2(1-2\eta^2)}{(2-2\eta^2+\theta)^{2.5}\;2(1-\eta)(1+\eta+\theta)^{1.5}}\right]V_\theta^2$$

Similarly the mean values $\bar{K}_c$ and $\bar{K}_c$ can also be defined by taking the mean value of each variable as

$$\bar{K}_s = \left[\frac{\bar{\theta}}{\alpha n(1+\eta)}\right]^{\frac{1}{n-1}}$$

Having known V_{kc} and V_{ks} , a criterion to find the buckling stress can be obtained from the equation (A.3) as

$$K_s^2\,(1-z^2\,V_{ks}^2)\;-2K_sK_c + K_c^2 - z^2\,V_{kc}^2\,K_c^2 = 0 \qquad (A.4)$$

The above equation is a function of θ only, that will determine the stress at buckling (refer Figs. 3.11-3.13 for intersection curves at various values of θ). For convenience, we have taken a constant coefficient of variation ($\bar{V}$) for all random variables. So solving (A.4) for the critical value θ_c at which buckling occurs, the buckling stress can be obtained as

$$K_s^* = \sigma^*\left[\frac{\theta_c}{\alpha n(1+\eta)}\right]^{\frac{1}{n-1}}$$

Fig. A-2 to A-7 give the variation in the buckling stress with σ^*. Here we point out that σ^* is a parameter which represents the material characteristics (σ_y, E) and also the thickness of the plate. For z = 3 (P_f = .00135) and for z = 5 (P_f=0.0000003), the buckling stress can be readily found, for variations of 0.01 and 0.05, 0.10.

The above values are compared with the corresponding values in Chapter 3 for σ^*= 2.5, 3 and 3.5 (Table A-1).

The equation (A.4) can also be used with advantage to probe the sensitivity of the reliability to the plate thickness (and hence on the cost of plate) for a given overall coefficient of variation and the hardening parameter θ, as shown in Figs. A-8 and A-9. Such curves are helpful to a designer, so that he can select a plate thickness, depending on the reliability sought.

TABLE A-1

COMPARISON OF THE CONVENTIONAL BIFURCATION STRESS WITH PROBABLISTIC STRESS FOR $\bar{V}=0.05, z=5$

η	$\sigma^*=2.5$		$\sigma^*=3.0$		$\sigma^*=3.5$	
	Conv.	Probl.	Conv.	Probl.	Conv.	Probl.
-0.9	3.6	2.15	3.77	2.20	3.90	2.22
-0.6	3.3	2.10	3.88	2.17	3.75	2.20
0.0	3.1	2.05	3.40	2.15	3.58	2.16
0.5	2.88	2.00	3.20	2.10	3.45	2.11
0.9	2.63	1.85	2.90	2.00	3.15	2.08

Conv. : Conventional method

Probl. : Probablistic method

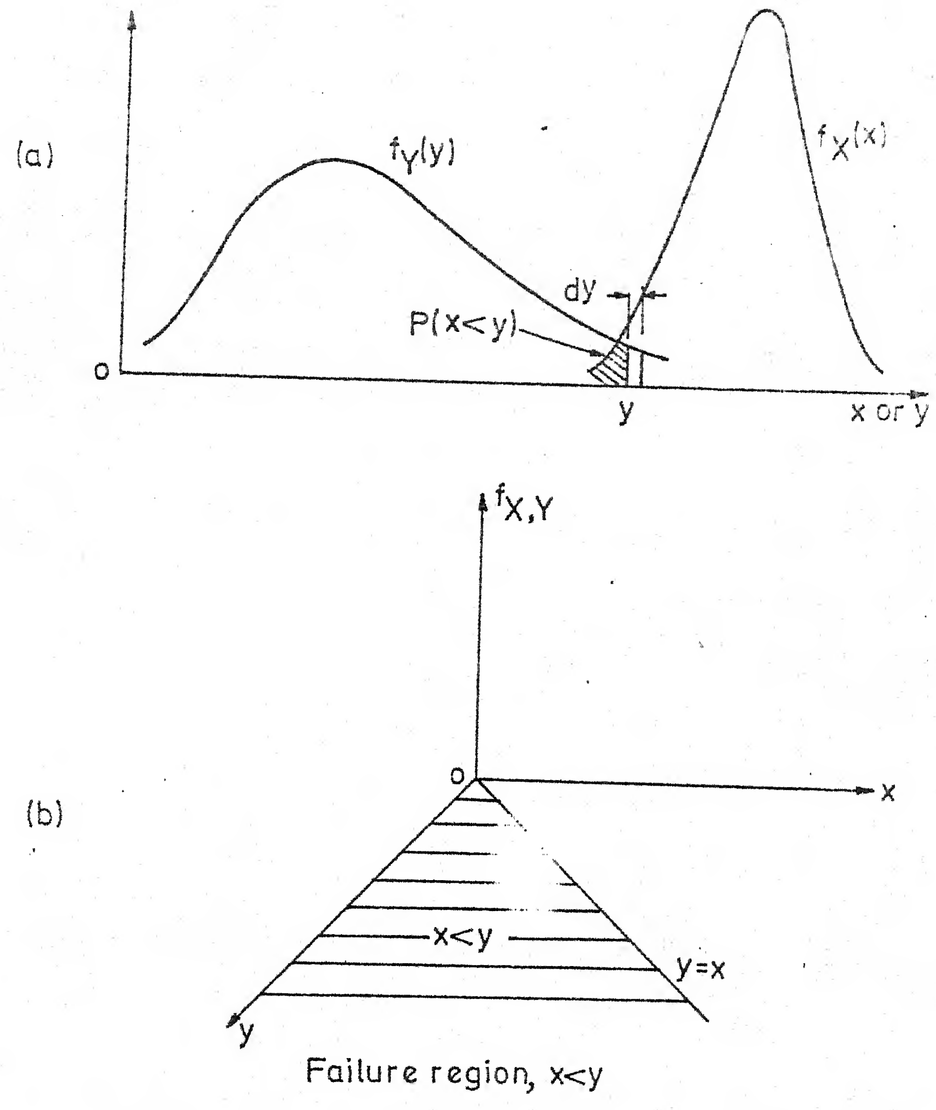


FIG. A-1 DENSITY FUNCTIONS OF LOAD AND RESISTANCE AND THE FAILURE REGION

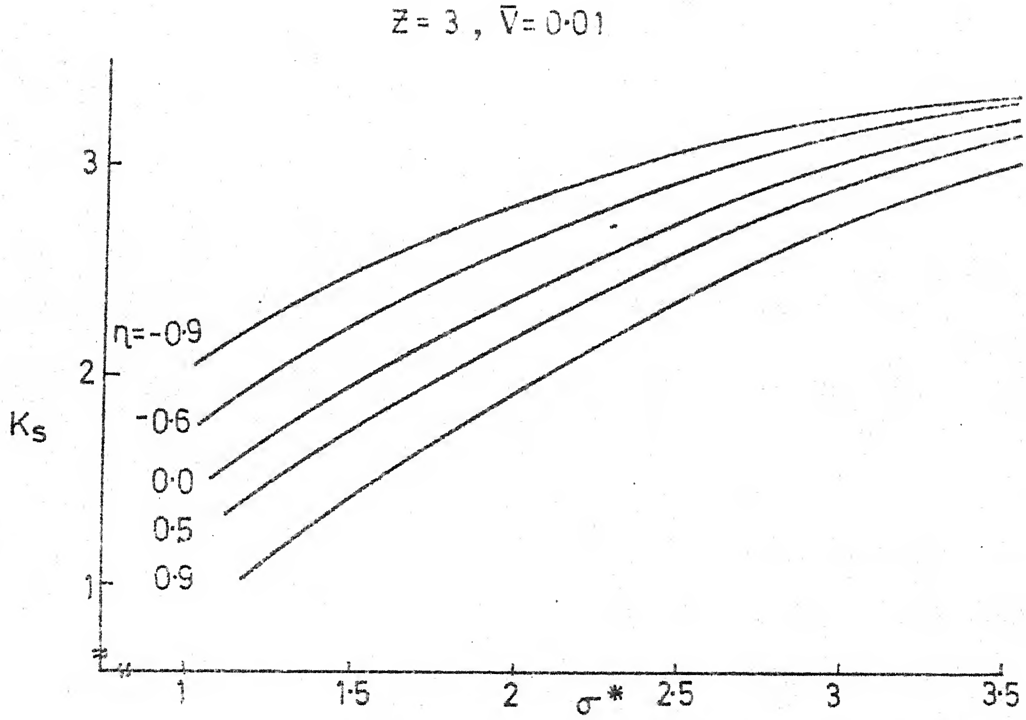


FIG. A-2 VARIATION IN BUCKLING STRESS WITH σ^*

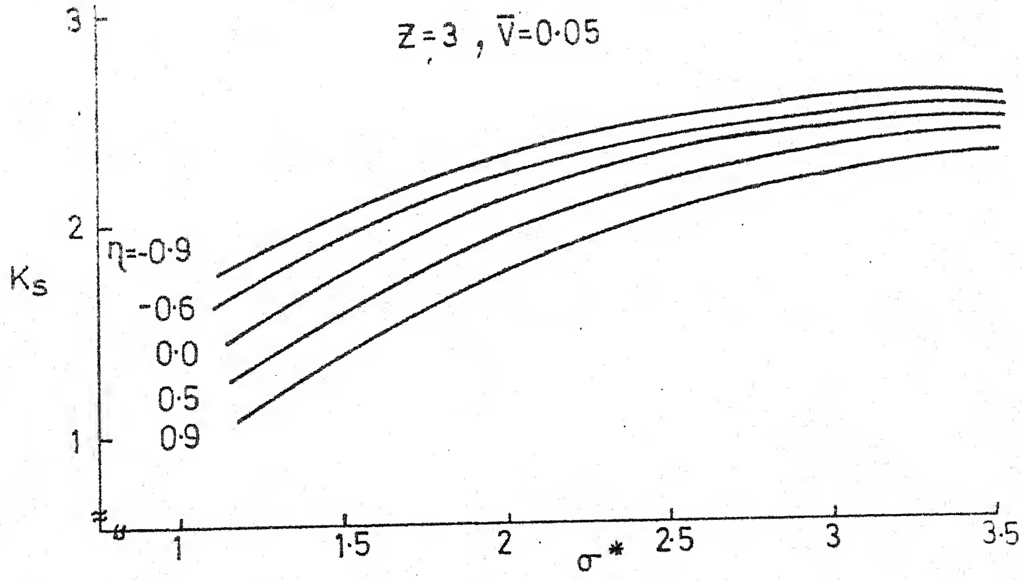


FIG. A-3 VARIATION IN BUCKLING STRESS WITH σ^*

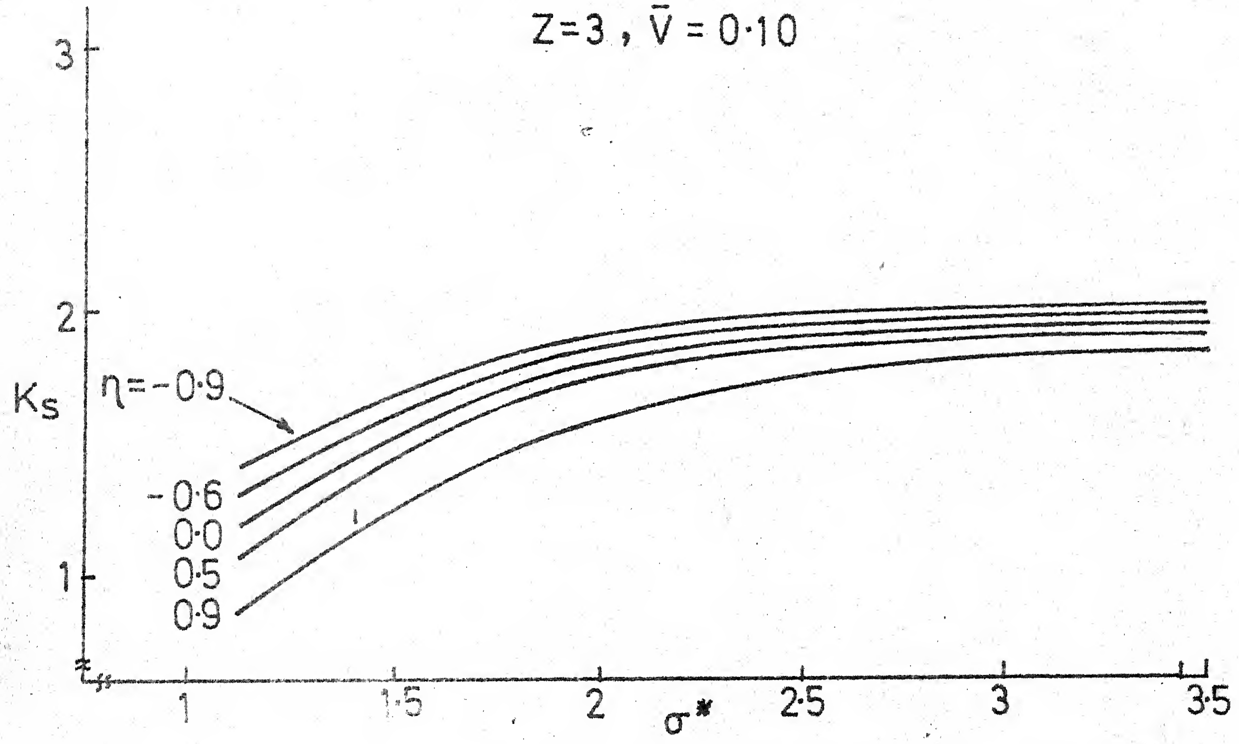


FIG. A-4 VARIATION IN BUCKLING STRESS WITH σ^*

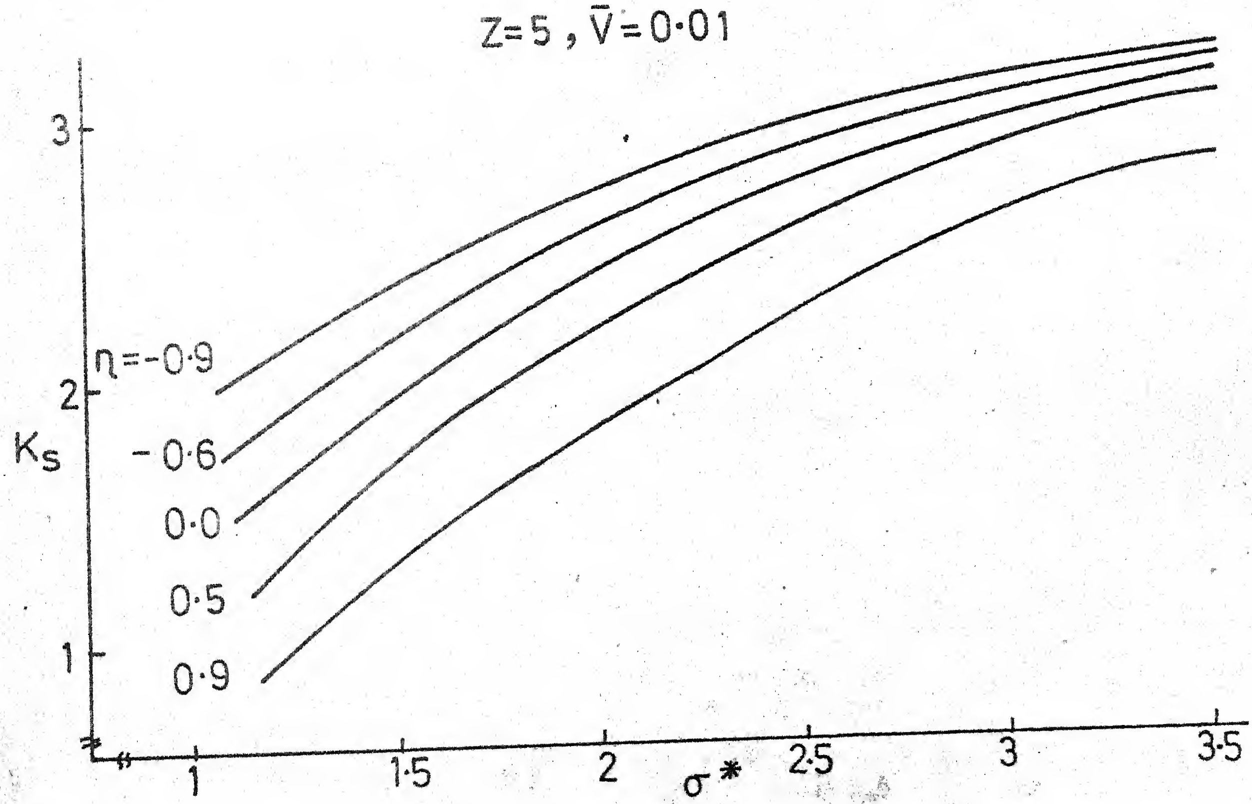


FIG. A-5 VARIATION IN BUCKLING STRESS WITH σ^*

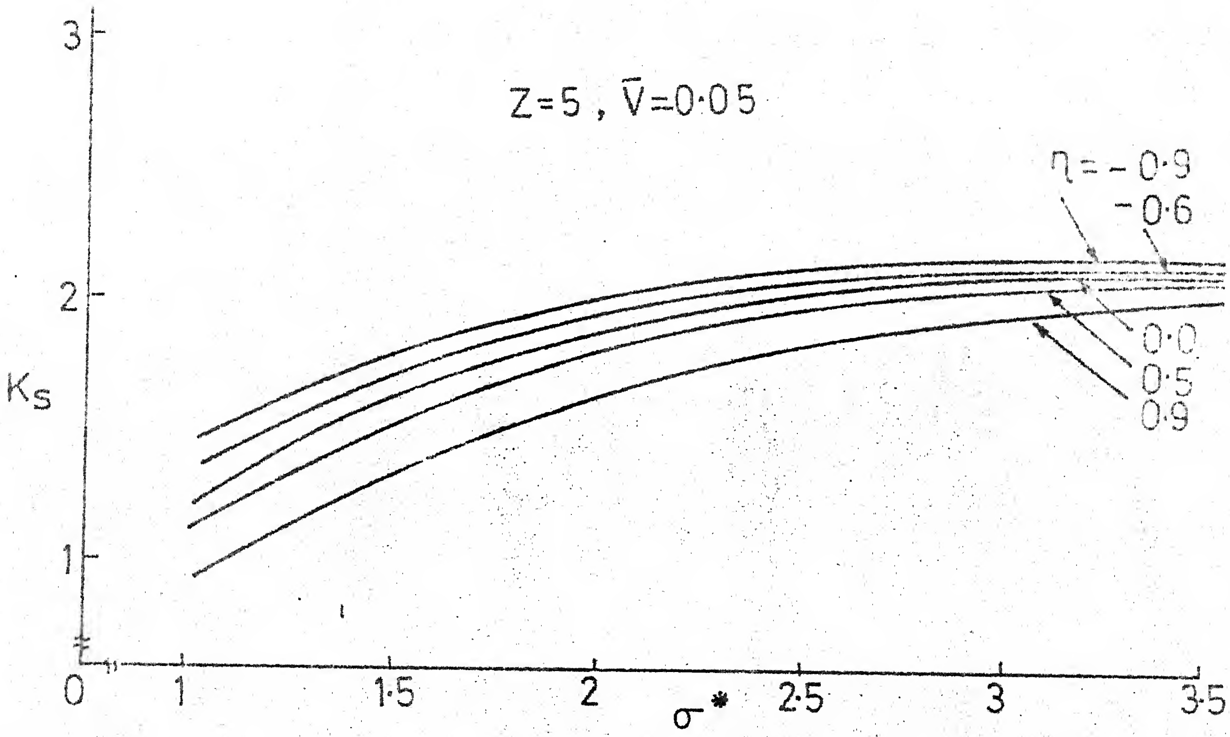


FIG. A-6 VARIATION IN BUCKLING STRESS WITH σ^*

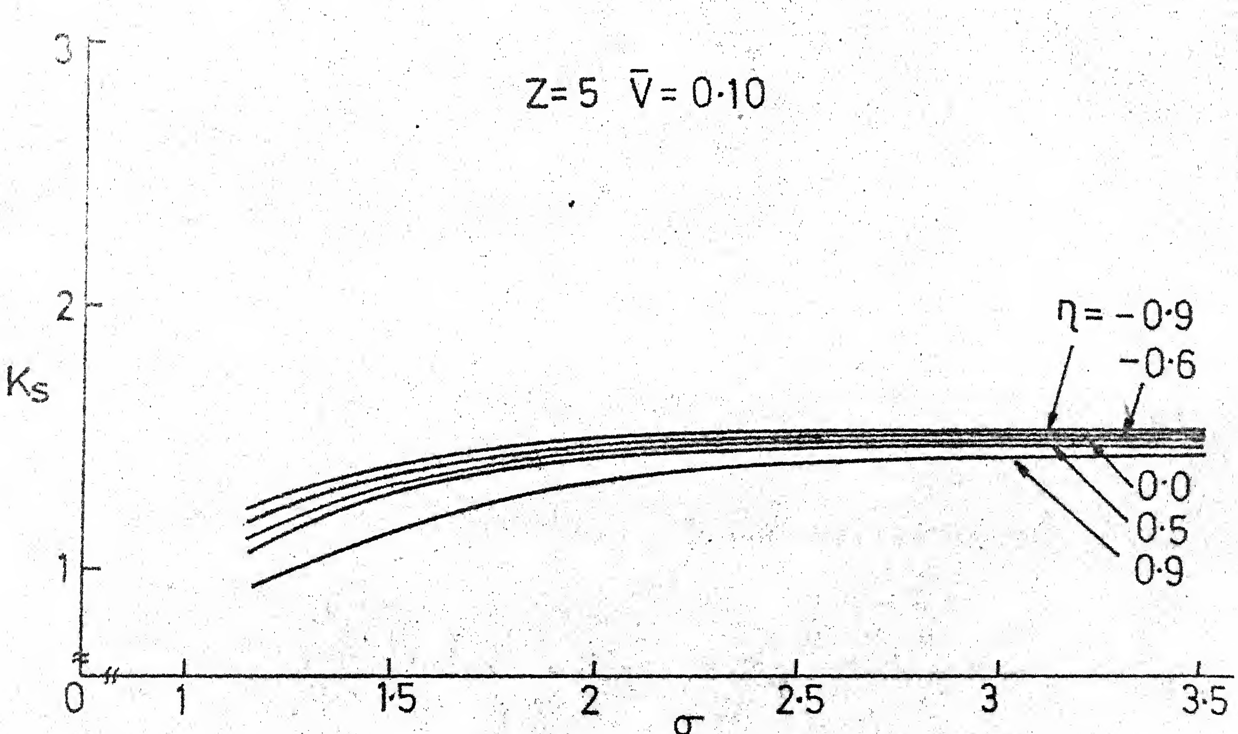


FIG. A-7 VARIATION IN BUCKLING STRESS WITH σ^*

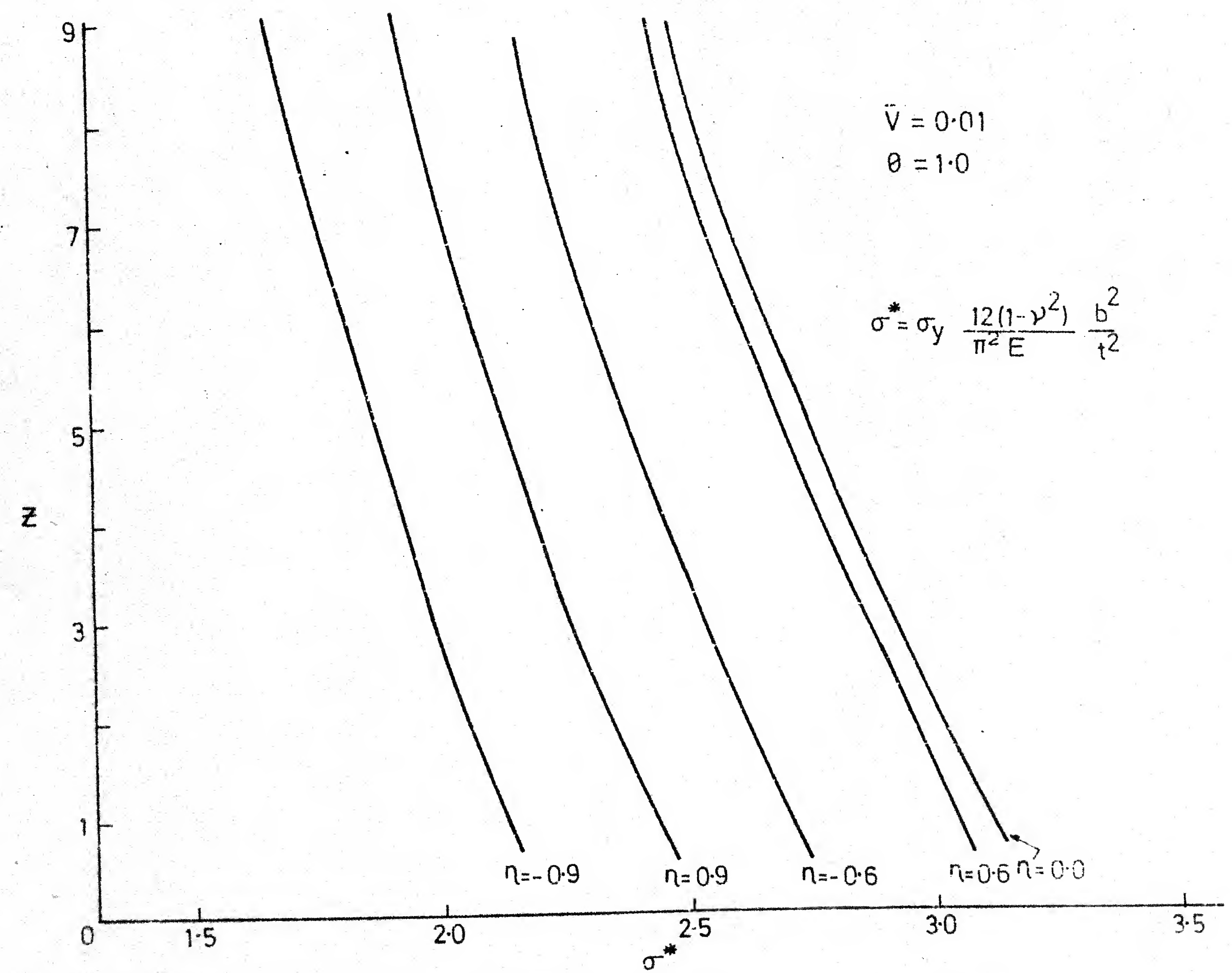


FIG. A-8 SENSITIVITY OF RELIABILITY TO GEOMETRIC TOLERENCE AND MATERIAL STRENGTH

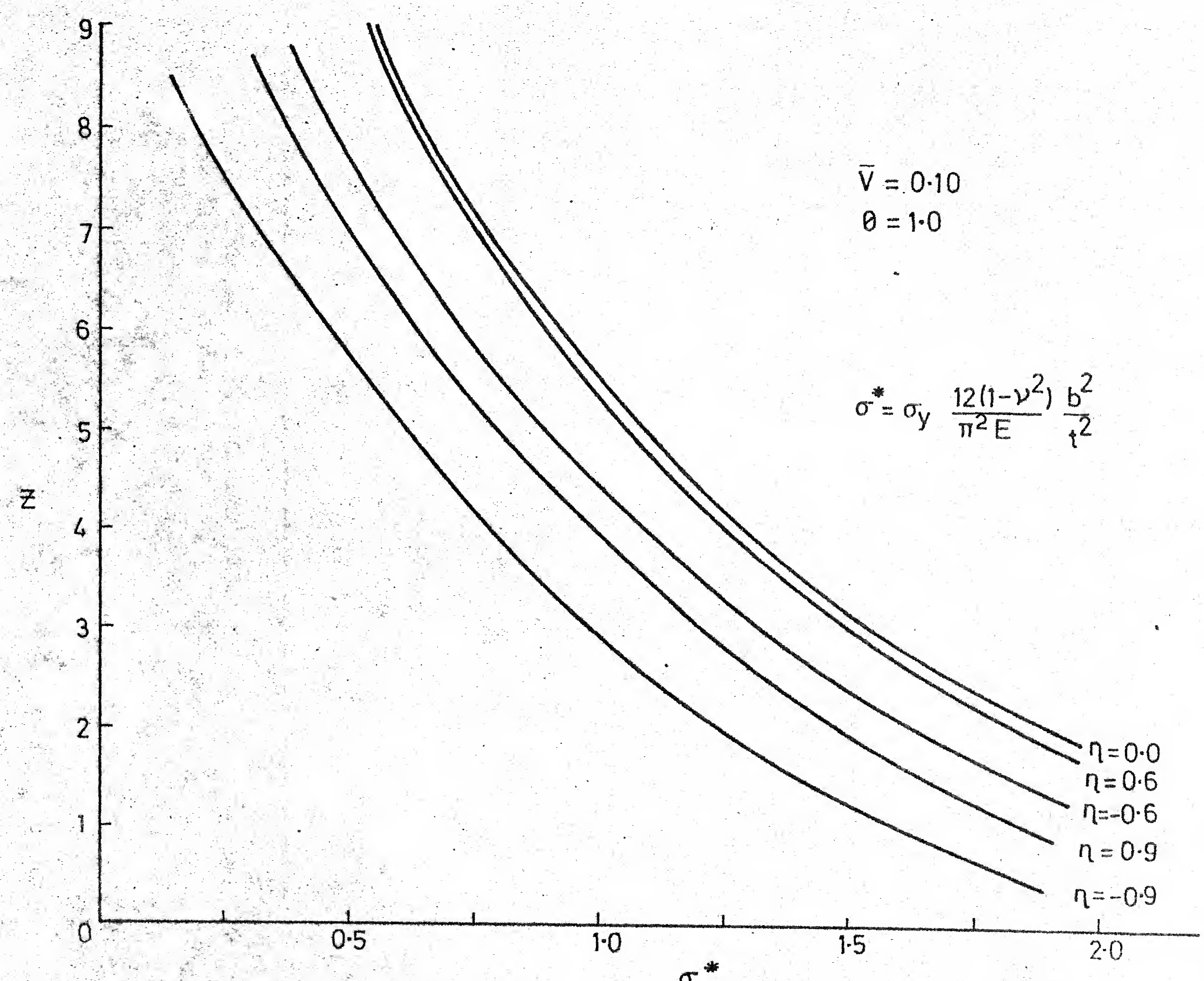


FIG. A-9 SENSITIVITY OF RELIABILITY TO GEOMETRIC TOLERENCE AND MATERIAL STRENGTH

APPENDIX B

STANDARD NORMAL PROBABILITY TABLE+

z	f(z)	Reliability R(z)	Probability of failure 'P_f'
0.0	0.398942	0.500000	0.500000
0.1	0.396952	0.539827	0.460173
0.3	0.381388	0.617911	0.382089
0.4	0.368270	0.655422	0.344578
0.6	0.333225	0.725747	0.274253
0.7	0.312254	0.758036	0.241964
0.8	0.289692	0.788145	0.211855
0.9	0.266085	0.815940	0.184060
1.0	0.241971	0.841345	0.158655
1.2	0.194186	0.884930	0.115070
1.4	0.149727	0.919243	0.080757
1.6	0.110921	0.945201	0.054799
1.8	0.078950	0.964069	0.035931
2.0	0.053991	0.977256	0.022750
2.2	0.035475	0.986097	0.013903
2.4	0.022395	0.991803	0.008197
2.6	0.013583	0.995339	0.004661
2.8	0.007915	0.997495	0.002505
3.0	0.004432	0.998650	0.001350
3.5	0.000873	0.999768	0.000232
4.0	0.000134	0.999968	0.000032
4.5	0.000016	0.999996	0.000004
5.0	0.0000015	0.9999997	0.0000003

+ From : Haugen, E.B., 'Probabilistic Approach to Design', John Wiley and Sons, New York, 1968.

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